Metastable Configurations of the Ising Model on the Torus

Julian Caballero Shinhae Park Hadrian Quan

Department of Mathematics
University of Illinois Urbana-Champaign

Program for Interdisciplinary and Industrial Internships at Illinois, Summer 2016

Outline

The Ising Model on the Torus

In this project, we investigated metastable configurations of the Ising model on the flat torus, a discrete model of magnetic spin.
The dynamics of the Ising model are determined by a Gibbs measure

$$\rho(\sigma) = \frac{1}{Z(\beta)} \exp(-\beta H(\sigma))$$

on the collection of states. Local minima σ of the functional $H(\sigma)$ will correspond to such metastable states, where

$$H(\sigma) = \sum_{(i,j)\in\Omega} \sigma_{ij}(2N_{ij}^{+} - 8),$$

and N_{ij}^{+} is the number of neighbors of the site (i, j) with value 1.

The Ising Model on the Torus

Some examples of metastable states on the flat torus include solid vertical and horizontal bands on the grid. We proved that is in fact the case:

Figure: Banded grid of active sites

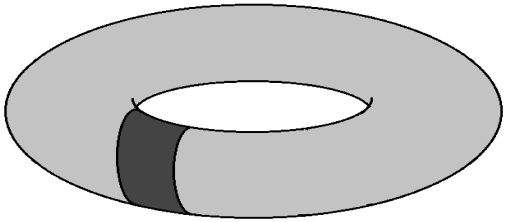


Figure: Metastable band on Torus

The Ising Model on the Torus

Theorem (Bands of width ≥ 3 are metastable)

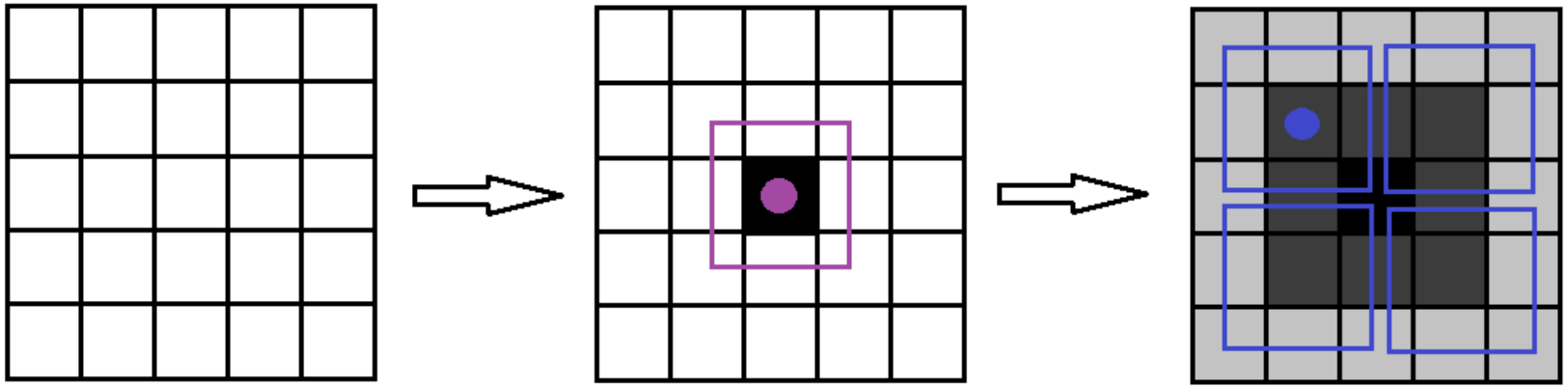
Let $\mathbb{T}_{m,n}$ be the flat torus, obtained from the quotient of a grid of size $m \times n$, with $m, n \geq 5$. If neighbors of a square in this grid are counted according to the von Neumann metric, then the following holds: if $\sigma \in \Lambda$ is a configuration given by a solid band of width ≥ 3, then for any other σ' which differs from σ by only one site,

$$H(\sigma) \leq H(\sigma'),$$

i.e. σ is a local minima of H.

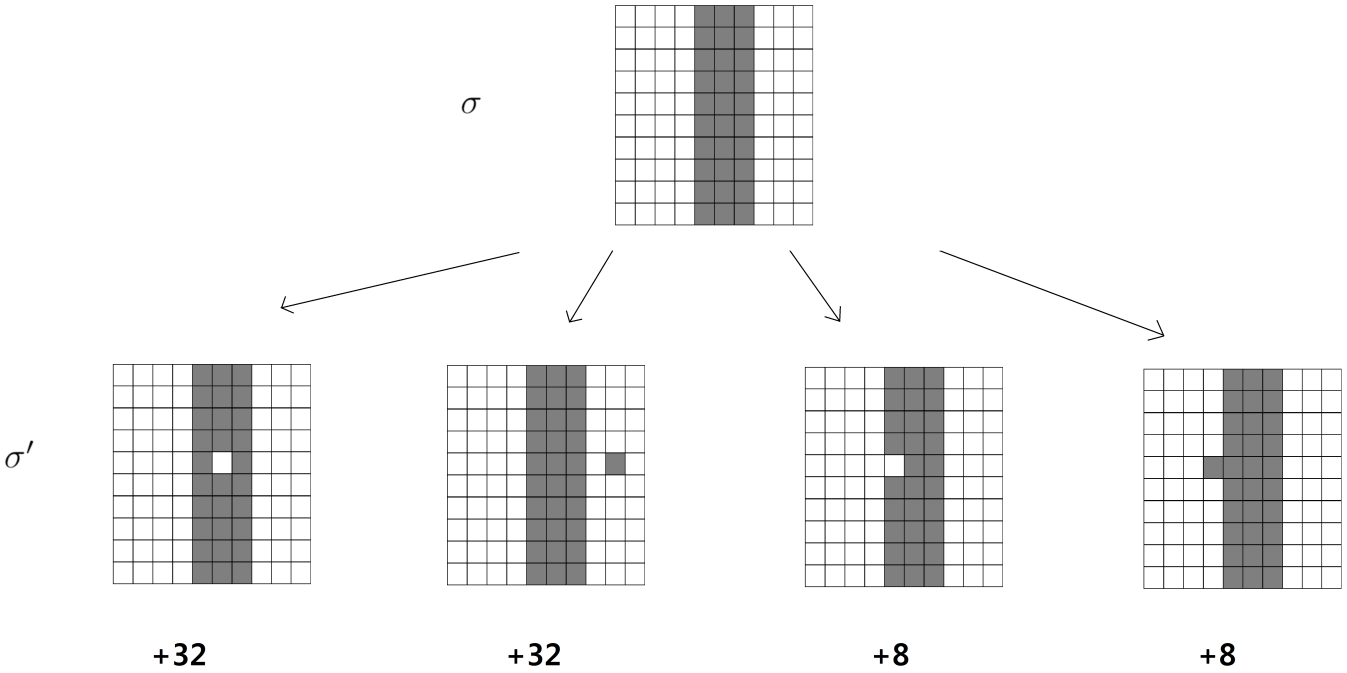
Metastability: Proof sketch

We can begin this argument with a reduction.



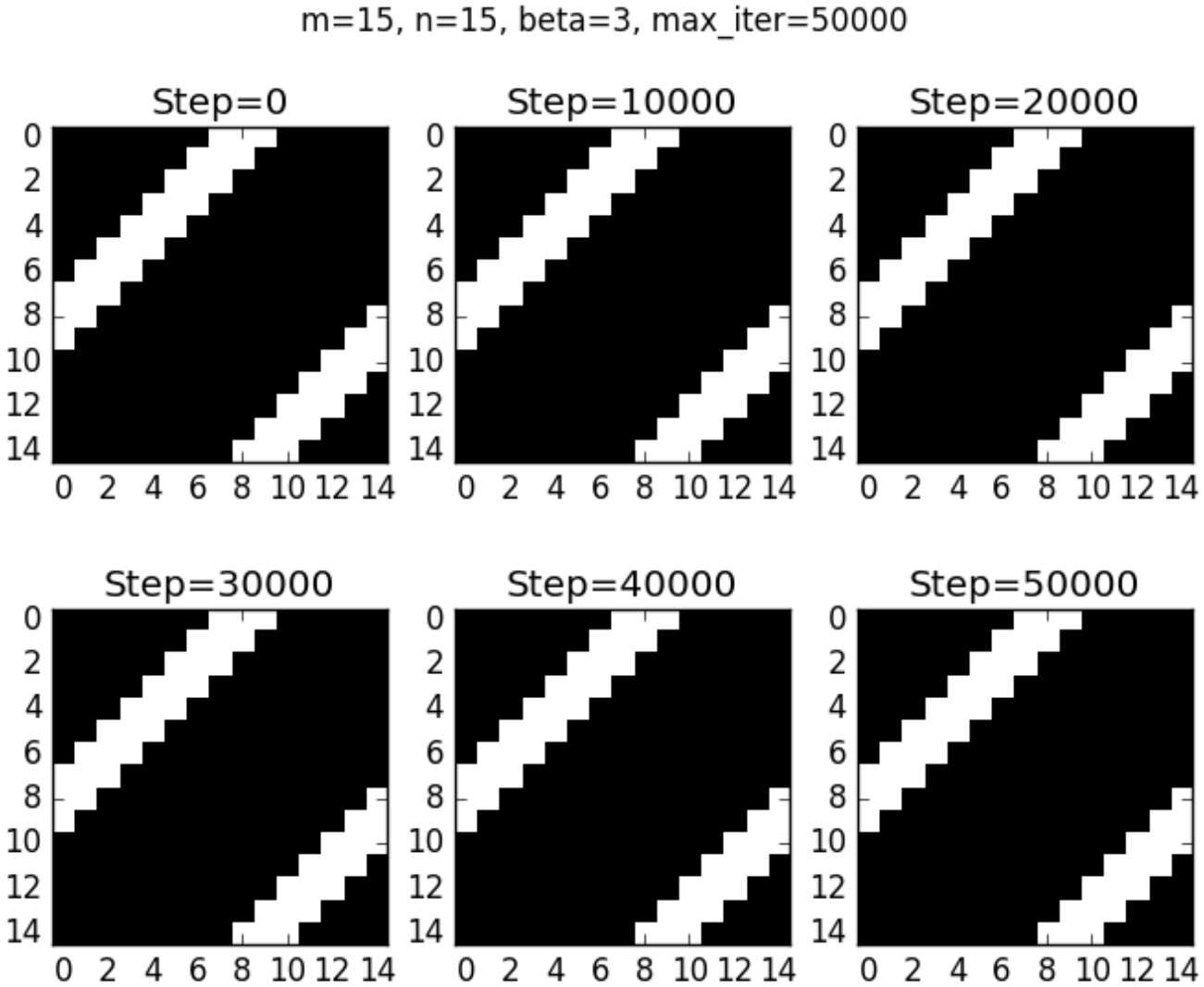
Proof cont'd.

The remainder of the proof proceeds by a direct computation in the 5×5 case.



Rare Metastable states

The previous argument shows that horizontal and vertical bands are metastable. However, these are not the only ones which can occur.



Metastable State Sampling

We had two quantitative concerns in this project.

- How should we determine whether or not the dynamics had fallen into a metastable state?
- How should we classify such metastable states?

These were addressed by: (1) using the conditional energy functional, and (2) using geometric intersection to compute homology classes of the bands.

Methods: Energy functional

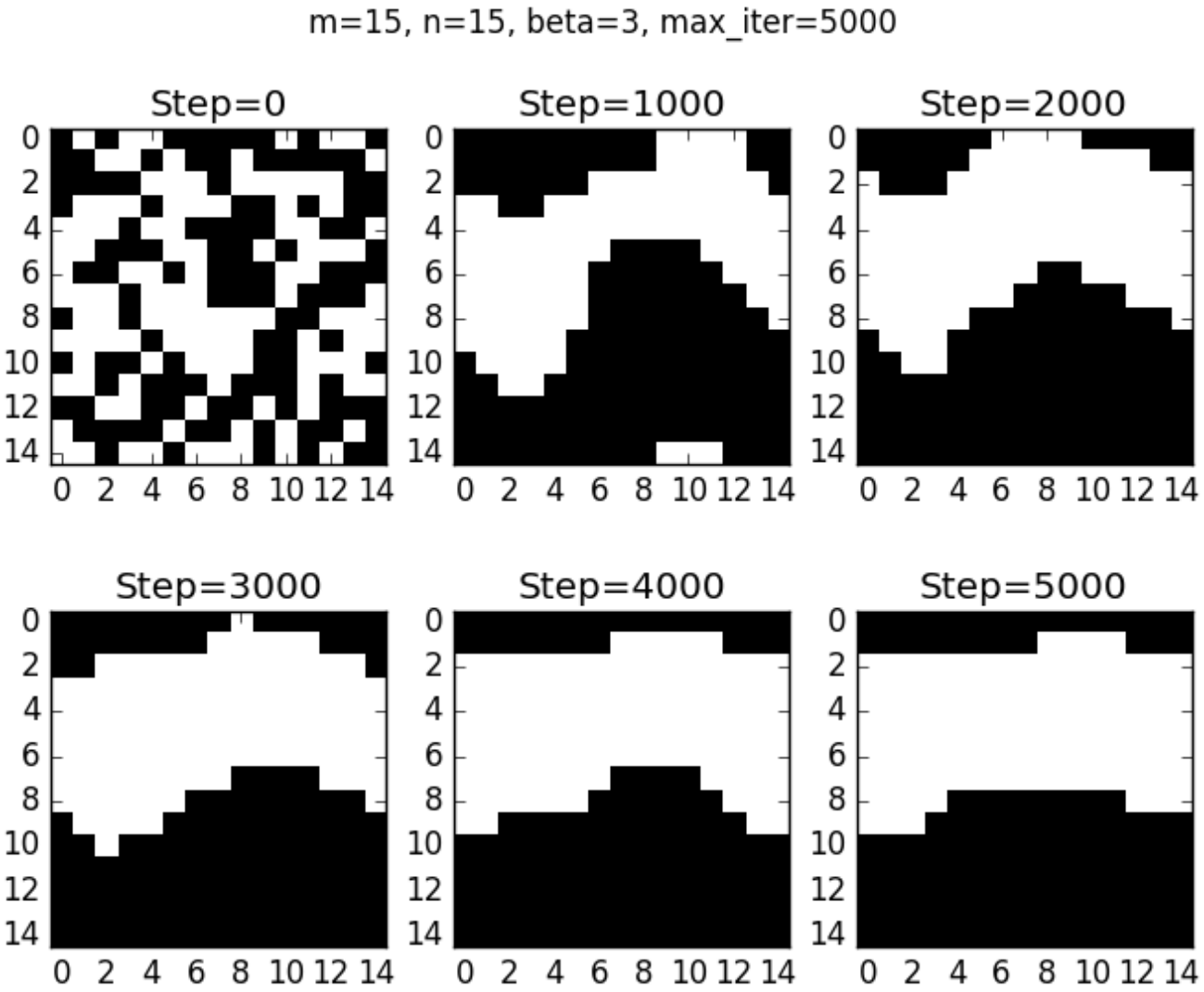
The (conditional) energy functional assigns a value of zero to the Stable States of our dynamics.

$$H(\sigma) = \sum_{(i,j)\in\Omega} 8 - \sigma_{ij}(2N_{ij}^{+} - 8)$$

$$\frac{\rho(\sigma)}{\rho(\text{All Aligned States})} = e^{-\beta H(\sigma)}$$

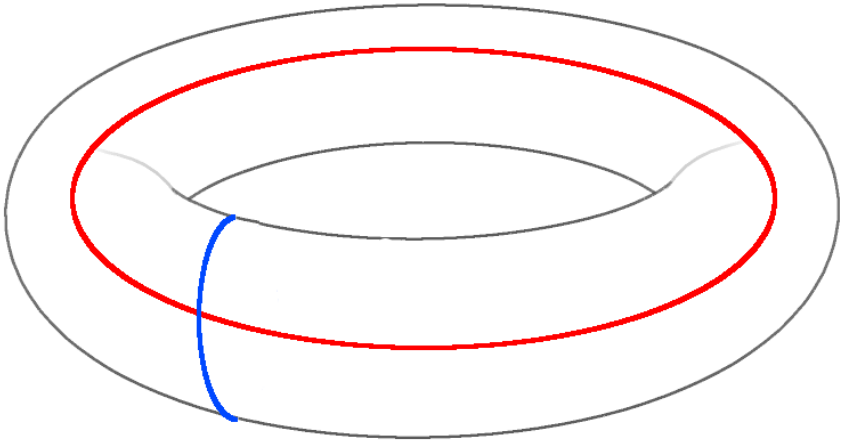
In our program, we periodically computed this quantity for the given state in the dynamics. When this quantity stabilized, it was a good indication of metastabiliy.

Methods: Energy functional



Methods: Geometric Intersection

The bands of active sites on our tori can be thought of as tubular neighborhoods of closed loops on the torus.



In our simulated grid, we marked two lines to represent generators for the homology classes of $\mathbb{T}^2$. By counting intersections of our bands with these generators, we could determine what elements in homology the corresponding loops should belong to.

Sampling Results

Array Shape (# samples)	#MS states	#Vertical bands	#Horizontal bands	#Diagonal bands
50x50 (10,000)	3556	1554	1591	394
60x30 (4,000)	2221	92	1973	72
20x60 (1,000)	708	651	1	1

Moduli Space of Flat Tori

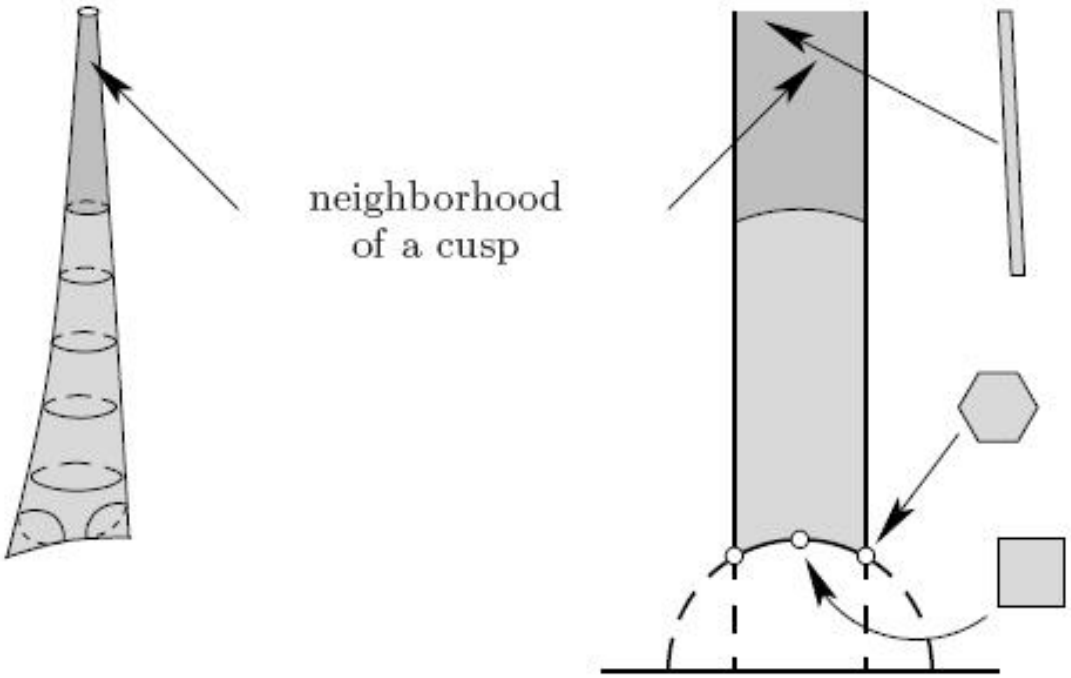
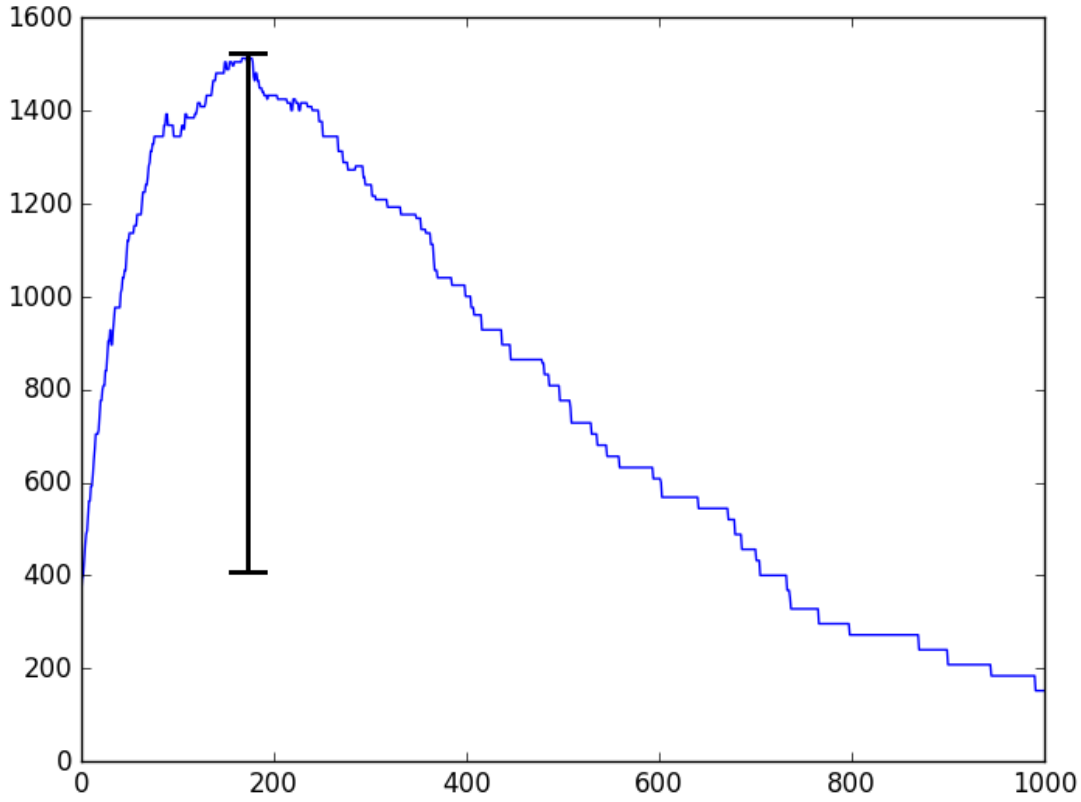


Fig. 13. Space of flat tori

The frequency with which metastable states on a given torus occur is proportional to the length of the corresponding closed geodesic it lies on in this moduli space.

Large Deviation Principle

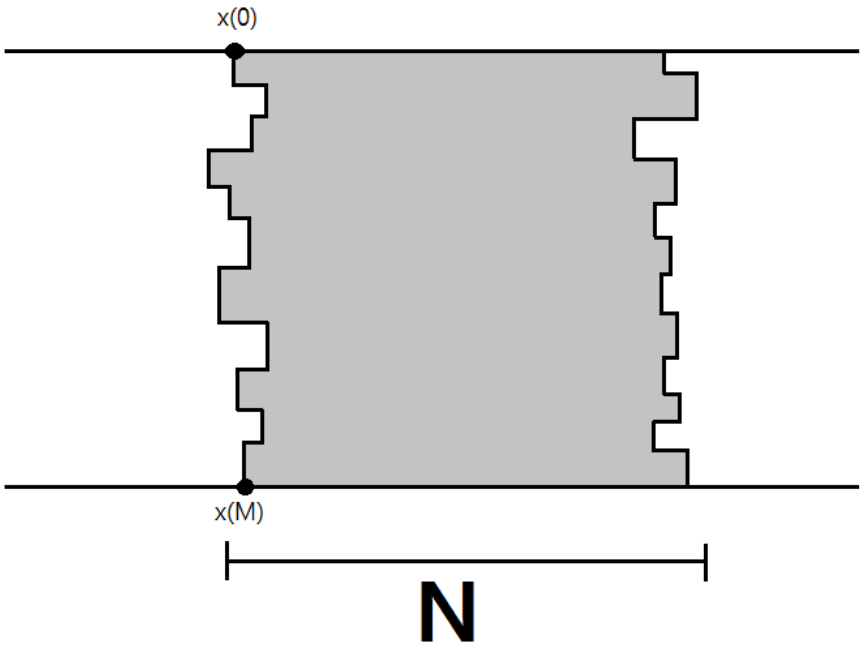
We've already considered, what is the probability to end up in metastable configuration?



A related question is: how long will it take to escape from metastable configuration?

Large Deviation Principle

Theorem(Mogulskii):
Simple symmetric random walk satisfies an LDP with good rate function.

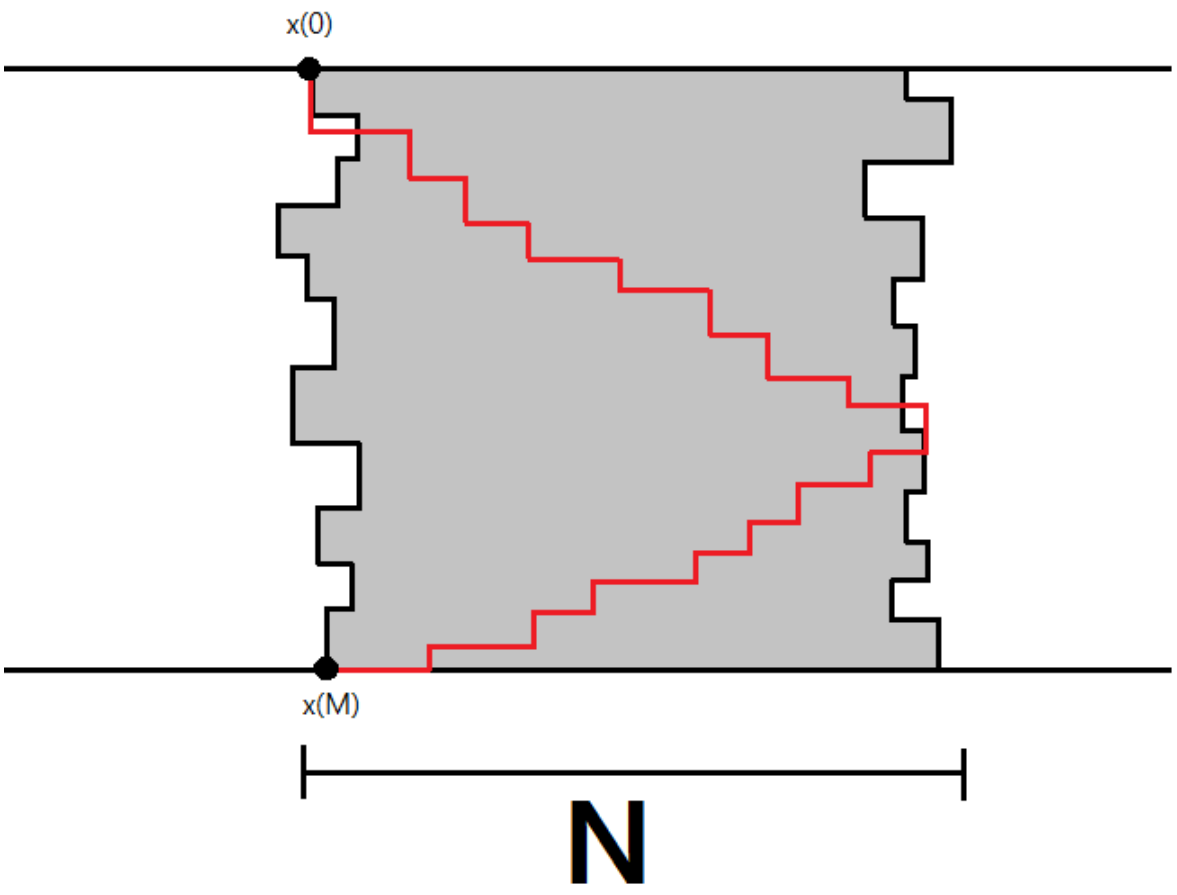


Want: Probability to cut a band will follow an LDP.

$$\mathbb{P}\left(\max_{t=0,\cdots,M} x(t)\cdot e_1 > N\right) \sim e^{-MJ(x(t))}$$

for some rate function J.

Random Walk



Any path which 'cuts' through the band can be thought of as part of a random walk. In dimension 2, random walk returns to its starting point has probability 1.

$e_1 = (1, 0)$

$$
\begin{aligned}
&\mathbb{P}\left(\max_{t=0,\cdots,M} x(t)\cdot e_1 > N\right) \\
&= \int_{\mathcal{P}} \mathbb{1}_{\{x(t)\cdot e_1 - N\}} d\rho(x(t)) \\
&= \int_{\mathcal{P}} \mathbb{1}_{\{x(t)\cdot e_1 - N\}} e^{-\beta(H(\sigma)-H(\sigma'))} dt \\
&\leq \int_{\mathcal{P}} e^{\alpha(x(t)\cdot e_1 - N)} e^{-\beta(H(\sigma)-H(\sigma'))} dt \\
&= \sum_{x\in\mathcal{P}} \sum_{t=0}^{M(x)} e^{\alpha(x(t)\cdot e_1 - N)-\beta(H(\sigma)-H(\sigma'))} \quad (*)
\end{aligned}
$$

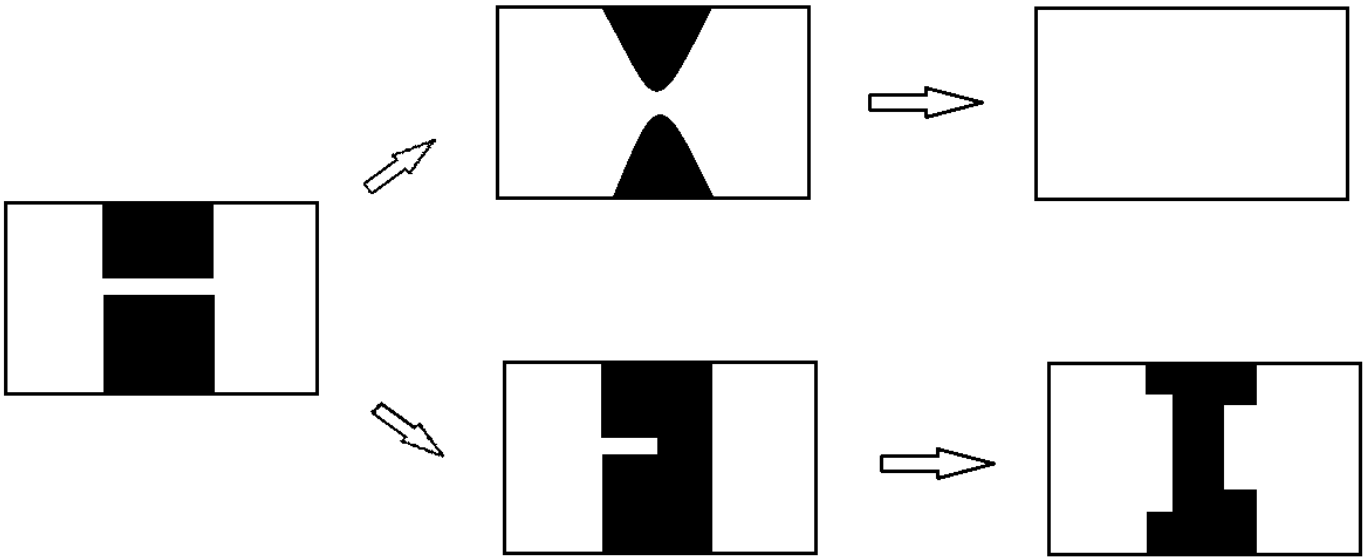
for any $\alpha > 0$.

Recall that we want:

$$\mathbb{P}\left(\max_{t=0,\cdots,M} x(t)\cdot e_1 > N\right) \sim e^{-MJ(x(t))}.$$

$$\begin{aligned}
&\mathbb{P}\left(\max_{t=0,\cdots,M} x(t)\cdot e_1 > N+1\right)\\
&\leq \sum_{x\in\mathcal{P}}\sum_{t=0}^{M(x)} e^{\alpha(x(t)\cdot e_1-(N+1))-\beta(H(\sigma)-H(\sigma'))}\\
&= \sum_{x\in\mathcal{P}}\sum_{t=0}^{M(x)} e^{\alpha(x(t)\cdot e_1-N)-\beta(H(\sigma)-H(\sigma'))}e^{-\alpha}\\
&= (*)e^{-\alpha}
\end{aligned}$$

Remark



Contrary to expectation, cutting a band may not lead immediately to a global minima.

For Further Reading I

D.A.Levin, Y.Peres, E.L.Wilmer
Markov Chain and Mixing Times.

G.B.Arous, R.CERF
Metastability of three dimensional Ising model on a torus at very low temparauters
Electronic journal of probability, 1(10):1–55, 1996.

A.Zorich
Flat Surfaces.
Frontiers in Number Theroy, Physics, and Geometry, 1(1):439–586, 2006.

Thanks

We are grateful for the support of Prof. Maxim Arnold and Prof. Yulij Baryshynikov throughout the course of this program. We also would like to thank Prof. Anil Hirani for his instruction in the Computational bootcamp.

PI4 is an NSF-funded award from the Division of Mathematical Sciences, in the Mentoring Through Critical Transition Points program; award 1345032.