Metastable States of the Ising Model on the Two-Holed Torus

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Outline

- Introduction
- Computational Results
- Theoretical Results
The L-shape

L-shaped grid as an identification space for the two-holed torus:
Squares of “hyperbolic type” – 16 neighbors, not 8
New definition of energy function – normalize!
Let \( v \) be a square on our L-shape.

\[
E(v) := -\frac{\sigma(v)}{N(v)} \sum_{w \in B(v), \ w \neq v} \sigma(w)
\]
What did we observe?

We saw some of these:
What did we observe?

And also some of these:
What did we observe?

And, surprisingly frequently, these:
Some more interesting configurations

Figure: Band configuration after 7,000,000 iterations.

Parameters:
- $m_1 = 60$
- $m_2 = 40$
- $n_1 = 60$
- $n_2 = 40$
- $\beta = 1.5$
(1 0 1 0)

(1 0 1 0)
Some Statistical results

<table>
<thead>
<tr>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Short bands</td>
<td>.413</td>
</tr>
<tr>
<td>Long bands</td>
<td>.064</td>
</tr>
<tr>
<td>Definitely diagonal bands</td>
<td>.062</td>
</tr>
<tr>
<td>Bands of type (0,1,0,1) or (1,0,1,0)</td>
<td>.148</td>
</tr>
<tr>
<td>Stable states</td>
<td>.250</td>
</tr>
<tr>
<td>Unclassified (1 1 1 1)</td>
<td>.003</td>
</tr>
</tbody>
</table>

This still leaves .040 bands (4% of them) unaccounted for because they have a ’2’ somewhere in the tuple
Theoretical Results
Local minima of the energy

Recall the total energy of a configuration:

\[ E(\tau) = - \sum_{v \in \mathcal{L}} \frac{\tau(v)}{N(v)} \sum_{w \in B(v) \setminus \{v_0\}} \tau(w) \]

- A **metastable** state is a spin configuration that persists for a long time.
- Such metastable states are local minima of the energy functional above with deep energy wells.
What is a local minimum of the energy?

Two spin configurations $\tau : \mathcal{L} \to \{\pm 1\}$ and $\sigma : \mathcal{L} \to \{\pm 1\}$ are said to be **nearby configurations** if $\tau$ and $\sigma$ only differ at one square.

A spin configuration $\tau$ is a **local minimum** if $\tau$ has a lower energy than all nearby configurations. Thus:

- Look at difference in energy:
  \[ \delta E(\tau, \sigma) := E(\tau) - E(\sigma) \]

- If $\forall$ nearby $\sigma$ we have $\delta E(\tau, \sigma) < 0$, then $\tau$ is a local minimum.
Are our observed metastable states local minima?
How to verify a local minima

Use the following expression for $\delta E(\tau, \sigma)$:

$$\delta E(\tau, \sigma) = 2\sigma(v_0) \sum_{\substack{v \in B(v_0) \\ v \neq v_0}} \sigma(v) \left( \frac{1}{N(v_0)} + \frac{1}{N(v)} \right)$$

$$= \frac{1}{8} \sigma(v_0) \left( (2H^+ + 3R^+) - (2H^- + 3R^-) \right)$$

where we’ve divided $B(v_0)$:

$$H^+ = \text{1’s of hyperbolic type} \quad R^+ = \text{1’s of regular type}$$

$$H^- = \text{-1’s of hyperbolic type} \quad R^- = \text{-1’s of regular type}$$
Example computation of $\delta E$

Here $H^+ = H^- = 0$, $R^+ = 3$, and $R^- = 5$, so:

$$\delta E(\tau, \sigma) = \frac{1}{8} \sigma(v_0) \left( (2H^+ + 3R^+) - (2H^- + 3R^-) \right)$$

$$= \frac{1}{8} (1)(-6)$$

$$< 0$$

Hence, $\delta E(\tau, \sigma) < 0$ so $E(\tau) < E(\sigma)$. 
Example computation of $\delta E$

One then looks at cases:
There’s a spin symmetry to the rescue

There’s a $\mathbb{Z}_2 = \{\pm 1\}$ action by multiplication on the set of configurations $\Omega$, where the action of $-1$ exchanges spins, i.e.

$(-1) \cdot \tau(v) = -\tau(v)$

The energetic difference for nearby configurations $\delta E$ is invariant with respect to this symmetry:

$$\delta E \left( (-1) \cdot \tau, (-1) \cdot \sigma \right)$$

$$= 2 \left( (-1) \cdot \sigma \right)(v_0) \sum_{\begin{subarray}{c} v \in B(v_0) \\ v \neq v_0 \end{subarray}} \left( (-1) \cdot \sigma \right)(v) \left( \frac{1}{N(v_0)} + \frac{1}{N(v)} \right)$$

$$= \delta E(\tau, \sigma)$$

This doubles our yield of local minima!
Our observations are local minima!

Lemma (Observed metastable states are local minima)

For a fine enough lattice graph $\mathcal{L}$, the following spin configurations are local minima of the total energy functional:

- **Thick enough straight bands** that are far enough from the hyperbolic point (a distance of 2 will do)
- **Thick enough diagonal bands** that are far enough from the hyperbolic point (a distance of 2 will do)
- Connected double bands
- One legged pants
- The set of squares of hyperbolic type
Some remarks and surprises

Why avoid the hyperbolic point in the first two examples? There’s a curious surprise that yields the answer...

It has a higher energy than the nearby configuration pictured!

\[
\delta E(\tau, \sigma) = \frac{1}{8} \sigma(v_0) \left( (2H^+ + 3R^+) - (2H^- + 3R^-) \right) = \frac{1}{8}(-1)(-9) > 0
\]
The problem is the band doesn’t have enough of the hyperbolic points (or too many depending on your point of view), which are a local minimum themselves!

The hyperbolic points are a local minimum and a strong one, as it displaces bands! This is part of the unique behavior of the Ising model on the genus 2 Torus.
The surprise demystified

Going back, we see that in our examples, the hyperbolic points are all one color:
Concluding remarks and Future Work

On the theoretical side:

- Explain the probability distribution of the observed homology types
- Estimate the probability of observing the metastable states by using large deviations
- Explain why straight and diagonal edges arise by using entropy pressure

On the computational side:

- One can vary the dimension parameters in our model and see its effect on the kinds of metastable states
- Implement other ways to study the frequency of the metastable states
References:

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