On the distribution of branching points of random polynomials

August 20, 2015

Julian Caballero, Robert Rennie, Mary Angelica Tursi, Lan Wang

1 Introduction

The content of this report mainly deals with the asymptotic behavior of branching points and their associated monodromy in the case of the bivariate polynomial. It is known that for a random bivariate polynomial

\[ f(x, y) = \sum_{j=0}^{n} \sum_{k=0}^{n-j} c_{j,k} x^j y^k \]

where \( c_{j,k} \sim N(0, 1) + N(0, 1)i \) with complex normal distributions for coefficients, the branching points cluster around the unit circle as the degree becomes increasingly large. In our work on branching points, we scaled the coefficients of a random polynomial with different scalings \( v_{j,k} \) to verify this behavior or to find cases in which there are exceptions. Our work on monodromy involved observing how the coefficient scalings affected certain monodromy distributions.

For the first part of our paper, we tried different scalings which resulted in different branching point distributions. Here we briefly mention seven different kinds of distributions. In each case, the arguments of the branching points were uniformly distributed, but radial distributions varied widely.

Case 1: \( v_{j,k} = k! \): Distributions consisted of many small rings of similar size distributed around the unit circle, with a small amount of points forming other shapes.

Case 2: \( v_{j,k} = \frac{k!}{j!} \): Distributions consisted of random rings of different sizes throughout the plane, though more concentrated in the origin. The distribution of the branching points was rather unstable, and the nature of the placement of the rings could not be determined, except perhaps a uniform distribution of the arguments of their "centers" and some increasing density of points closer to the origin.

Case 3: \( v_{j,k} = \frac{a}{j!} \) and \( v_{j,k} = \frac{a}{(j+k)!} \), (\( a \) =large scale): Distributions consisted of a filled circle of a certain radius. As the degree increased, so did the radius.
Case 4: \(v_{j,k} = a \ast (j! + k!)\) and \(v_{j,k} = a \ast (j! \ast k!)(a=\text{small scale})\): Distributions consisted of multiple circles with different radii inside unit circle.

Case 5: \(v_{j,k} = j!k!(n - j - k!):\) A special form of convergence to the unit circle, this consisted of \(n\) rings uniformly distributed at or near the unit circle.

Case 6: \(v_{j,k} = p(j, k, n), p(j, k, n)\) is polynomial respect to \(j, k, n\): These distributions resembled that of identically distributed coefficients. The branching points clustered near unit circle.

Case 7: \(v_{j,k} = \frac{n!}{j!(n - j - k)!}\) and \(v_{j,k} = \frac{n!}{(j + k)!(n - j - k)!}\): This was one of the most stable, non-unit circle cases. The placement of branching points became more dense as they approached the origin, and were increasingly sparse as they branched out. There was not a dense ring around the unit circle.

Among these cases, we further explored the last two cases, and made conjectures which will be discussed later.

For the second part of our paper, we briefly discussed an extension of our work on branch point distributions in which we explored the distributions of their associated monodromy group elements.

2 Variation from polynomials’ scale

It is well known that the branching points of a random bivariate polynomial cluster around unit circle as the degree approaches infinity. [3] As shown above, this distribution could be altered with even a tiny variation of the polynomial’s coefficients. An ambitious problem is to determine a sufficient condition for the variation such that the branching points of an random polynomial with such variation keeps the “unit-circle distribution”.

In this paper, we do not aim to solve this problem, but to provide some inspiration by showing a particular type of variation which keeps the “unit-circle distribution”. We will begin with a conjecture, and then support it by illustrating our experimental work and simple theoretic analysis.

2.1 Conjecture 1

For a bivariate polynomial \(f(x, y) = \sum_{j=0}^{n} \sum_{k=0}^{n-j} c_{j,k} x^j y^k\) of degree \(n\) with random uniform coefficients, if scaling the coefficients with polynomial variation \(v_{j,k}\), where the powers of \(x\) and \(y\) are \(j\) and \(k\), respectively, that is, altering \(f\) such that,

\[f(x, y) = \sum_{j=0}^{n} \sum_{k=0}^{n-j} v_{j,k} c_{j,k} x^j y^k,\]

then the branching points will still cluster near unit circle as the degree \(n\) approaches infinity.
2.2 Experiments

In order to experimentally check the validity of our first conjecture, we conducted experiments in MATLAB.

2.2.1 Experimental steps

We organized the experiments into several steps listed as follows:

1. Generate an \((n+1)(n+2)/2\)-by-1 array of Gaussian integers whose real and imaginary parts are both uniformly distributed from \(-100\) to \(100\). The elements of the array are the coefficients of a bivariate polynomial \(f(x,y)\) of degree \(n\).

2. Scale the coefficients with polynomials with respect to \(n, j\) and \(k\), such as \(j + 1, k + 1, j + k\) or \((n + 1 - j) \times (n + 1 - k)\).

3. Plot the distribution of \(n(n-1)\) branching points of the scaled polynomial on complex plane.

4. Fix each scale, and put the degree \(n\) as variable. Calculate the percentage of branching points which cluster near unit circle \((0.8 < radius < 1.2)\), and draw a pie chart to roughly describe the distribution.

5. Gather data and draw the conclusion.

2.2.2 Result

In general, as the degree \(n\) increases, a greater percentage of branching points cluster near unit circle with radii between 0.8 and 1.2. Take \((n + 1 - j) \times (n + 1 - k)\) as our representative example. The highest degree \(n\) of polynomial that we tested without sacrificing data accuracy is 25. As \(n\) ranges from 5 to 25, the percentage of branching points lying near unit circle increases from 25% to 67% as shown in Table 1 and Figure 1 below:

<table>
<thead>
<tr>
<th>degree (n)</th>
<th>Percent around unit circle</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>25%</td>
</tr>
<tr>
<td>10</td>
<td>37%</td>
</tr>
<tr>
<td>15</td>
<td>50%</td>
</tr>
<tr>
<td>20</td>
<td>62%</td>
</tr>
<tr>
<td>25</td>
<td>67%</td>
</tr>
</tbody>
</table>

Table 1

When \(n = 25\), the “near-unit distribution” feature is apparent, as shown in Figure [I].
2.3 An idea of a theoretical proof

Inspired by D’Andrea, Galligo, and Sombra’s work [1], we consider the critical points of the curve $f(x, y) = 0$ instead of considering the branching points directly (since branching points are just projections of critical points):

To find the critical points, all that is needed is to solve the following system of equations:

$$\begin{cases} f(x, y) = 0 \\ \frac{\partial f(x, y)}{\partial y} = 0 \end{cases}$$
From a previous study [1], we know that the roots of a system of independent equations
\[ f_1 = f_2 = \cdots = f_n = 0 \]
are “approximately equidistributed near the unit polycircle” under some specific conditions of the coefficients of the \( f \)’s. In our case, the two equations relate very closely to each other so that we expect much looser conditions to make our critical points cluster near unit torus. This suggests that the branching points cluster near the unit circle (since, again, branching points are projections of critical points to the \( x \)-plane). Once we find such conditions and determine that the coefficients with polynomial variations satisfy them, **Conjecture 1** will be proven.

### 3 Exceptions to the Unit Circle

As mentioned before, scaling the coefficients of a polynomial has an effect on the asymptotic behavior of univariate polynomials of increasing degree. An example of this can be found in Edelman and Kostlan’s work with random polynomials with real coefficients. [2] If the coefficients of a polynomial of degree \( n \) are identically distributed, then the portion of real roots is of order \( \log n \). However, when scaling coefficients by \( i^n \), where \( i \) is the degree of the term, the portion of real roots approaches \( \sqrt{n} \). That such a phenomenon is influenced by the mere scaling of coefficients suggests to us that distributions of roots may change according to various scalings.

In many such cases, we found that real polynomials differed little from their complex counterparts with respect to radial distributions and placement on the plane, even though angular distributions were of course affected and the resulting image was symmetric about the real axis. However, we often resorted to testing real polynomials first because it was computation-wise far easier, both in speed and in size of degree, and then tested the case for bivariate polynomials of smaller degree, where the coefficients had normal complex distributions. Furthermore, it was often necessary to multiply the polynomial by some scalar to avoid both coefficients smaller than machine epsilon or those which are too large for computation. Some scalings (see cases 1-5) could only be tested on polynomials of at most degree 20, whereas others (see cases 6,7) had coefficients tame enough for working with polynomials up to degree 30 or so.

In order to get a greater sense of possible curvatures of the distribution of radii, we took the natural logarithm of radial lengths, and examined the mean, median, and variance of the logs. A branching point distribution clustering around the unit circle would have radial distributions with a mean approaching zero and decreasing variance. Thus if either there is an increasing variance, or the mean does not seem to approach zero, we can be confident that the branching points are not approaching the unit circle. This method also worked well with assessing the stability, or lack thereof of said distributions, and was helpful in distinguishing the different types of exceptions thus far encountered. Most importantly, it gave insights into the actual distributions on branching points.

Here we shall look at two distinct but nonetheless similar cases, where the branching points do not appear to converge to the unit circle. The scalings are based on case 7 mentioned above in the introduction.
3.1 Case 7a: \( v_{j,k} = \frac{n!}{j!k!(n-j-k)!} \)

MATLAB simulations were reliable up to \( n = 33 \). The placement of branching points became more dense as they approached the origin, and were increasingly sparse as they branched out. There was not a dense ring around the unit circle. Rather, as the degree increased, the concentration was more focused towards the origin, but there were significantly many points with radial lengths far larger than those found in ”unit-circle” distributions.

![Figure 3: Left: branching points for one real polynomial with scaling \( v_{j,k} = \frac{n!}{j!k!(n-j-k)!} \), where \( n = 30 \). Right: branching points for 10 polynomials with complex coefficients with scaling \( v_{j,k} = \frac{n!}{j!k!(n-j-k)!} \), where \( n = 20 \)](image)

Not only did log distributions reveal relative stability for polynomials of different degrees, it also pointed certain concrete characteristics of the radii themselves, which leads to the following conjectures:

**Conjecture 2.1**

Let \( f(x, y) \) be a random bivariate polynomial with complex normal distributions for coefficients with the scaling \( v_{j,k} = \frac{n!}{j!k!(n-j-k)!} \), where \( n \) is the degree of the polynomial, and \( j \) and \( k \) are the degrees of \( x \) and \( y \), respectively. Then the distribution of radii is radially symmetric around the unit circle as \( n \to \infty \).

This is supported by taking the logarithms of the radii. The distributions of the logs reveal a median around 0, and are also symmetric around the origin. See figures 4 and 5 below.

A similar visualization of the points supports this conjecture, when the radii of points are plotted with their multiplicative inverses. The following is an example with a polynomials of degree \( n = 25 \) and \( n = 30 \). See figure 6:

**Conjecture 2.2**

For each polynomial degree \( n \) with the characteristics described in Conjecture 1, the distributions of the radii approach a log-normal distribution, that is, for a branching point \( p \), \( \log |p| \sim N(0, \sigma^2) \) with increasing \( \sigma \) as \( n \to \infty \).

Repeated simulations revealed stable distributions of branching points for polynomials of each degree \( n \). However, as can be seen in the plot below, the increase in variance of the logs
Figure 4: Top: log-radial distribution of 15 samples of branching points of polynomials with real coefficients, where $n = 15$. Here the best fit line is computed through MATLAB itself, whereas the sample data curve is derived by directly calculating the mean and variance of the points.

(a) Logs of radii for a single polynomial with complex coefficients of degree $n = 20$.

(b) Logs of radii for a single polynomial with real coefficients of degree $n = 30$.

Figure 5: Higher degree polynomials tend to stabilize on their own, whereas distributions of lower degree polynomials require more samples to be seen clearly.

of the radii leaves open the question of whether the overall stability of distributions remains as $n \to \infty$.

The log-normal distribution of radii could be better understood if the univariate case were studied with the analogue scaling $n$. This scaling revealed a similar behavior for the univariate case. Like the bivariate case, the radii seem to be log-normally distributed with a median of 1 (corresponding to the log-radius length 0). Below is a sample of the logs of the radii and their distributions. For univariate polynomials. See figure 8.

In addition the variance of the distributions of the logs of radii are increasing. It remains unknown whether or not the variance approaches a limit. See figure 9 below.
Figure 6: Left and Right, respectively: polynomials of degree 25 and 30 with radial length of branching points and their inverses in order

Figure 7: variances ($\sigma$) of distributions for logs of radii of the branching points for polynomials of increasing degree

3.2 Case 7b: $v_{j,k} = \frac{n!}{(j+k)!(n-j-k)!}$

Distribution on the onset appeared similar to that of case 7a, though there was less density inside of the unit circle. MATLAB simulations were reliable up till $n = 25$. This case showed less stability than Case 7a polynomials, since the median was also increasing. See figure 10.

However, there also seemed a fixed percentage of points, close to 30% inside the unit circle. This can be more clearly seen, once again, by taking the logs of the radii. The above leads to some analogous conjectures for case 3b:

Conjecture 2.3

Let $f(x, y)$ be a random bivariate polynomial with complex normal distributions for coefficients with the scaling $v_{j,k} = \frac{n!}{(j+k)!(n-j-k)!}$, where $n$ is the degree of the polynomial, and $j$ and $k$ are the degrees of $x$ and $y$, respectively. Then the following limit exists:

$$
\lim_{n \to \infty} \frac{\#\{p : |p| < 1, \ p \text{ is a branching point}\}}{\#\{p : p \text{ is a branching point}\}}
$$

As can be seen below, the following plots for the cdf of radii of polynomials of varying
Figure 8: Left and Right, logs of radial lengths of the roots of univariate complex polynomials of degrees 20 and 200, with scaling \( \binom{n}{i} \), for coefficient with term of degree \( i \).

Figure 9: Variance of logs of radial lengths for complex polynomials of degree 5-200. One thing to note, as seen in Figure 8, is that the difference in \( \sigma \) calculated by the best fit method, and calculating the direct variance, becomes more disparate. It is not known exactly why that is the case, though it may be that directly computing \( \sigma \) from the data may better take into account the outlier points, whereas best fit methods will be produce results skewed towards the center.

degree support the conjecture. Yet while not radially symmetric, there does seem to be a symmetry around the actual median for the logs of radii of the branching points of \( n \)-degree polynomials. See figure 10 for a log distribution. So we also have the following conjecture:

**Conjecture 2.4**

*For each polynomial degree \( n \) with the characteristics described in Conjecture 1, the distributions of the radii approach a log-normal distribution (i.e., for a branching point \( p \), \( \log |p| \sim N(\mu, \sigma^2) \)), with increasing \( \mu, \sigma \) as \( n \to \infty \).*

At this point it is even less certain whether the limiting behavior of the branching points is stable, as both the median and variance of the logs is changing as \( n \) increases, as shown in figure 11.
Figure 10: Left, Middle, Right: logs of radial lengths of branching points of polynomials of degrees 10 (25 samples), 15 (15 samples), and 20 (6 samples), with scaling $(\frac{n}{j+k})$.

Figure 11

4 Distributions of Monodromy Group Elements

A natural extension of our problem is to look at the distributions of elements of the monodromy groups corresponding to the polynomials’ branch points. By ordering the branching points by their arguments we get an ordering of the permutations associated to them, and thus a well-defined problem (in the generic case). Unfortunately, this problem proves to be quite difficult, as computational limits prevent any meaningful experimental intuition.

We first collected data on the case of random, degree-three, bivariate polynomials with coefficients taken from the normal distribution, then we took data on the same distribution with coefficients scaled as in case three above. With the help of the algcurves maple package, the two data sets were collected by two computers working nonstop for over a week. The raw data and the code used to produce it can be found by clicking here. In Figure 13, we have taken the raw data for the normal case, and counted the occurrences of monodromy element sequences up to relabeling of the monodromy elements and the action of cycling their order. The plots in Figure 12 are of these counts in descending order to give an idea of the distribution.
Figure 12

(a) A plot of the normal data in descending order.  (b) A plot of the data for the scaled case.

Figure 13: The frequency of representatives of orbits of monodromy element sequences under relabeling of elements and cyclic permutation of sequences.

5  Conclusion

In this paper, we explored the distributions of branching points of random bivariate polynomials with complex normal distributions for coefficients with different scalings. We basically divided these distributions into 7 cases. Among them, two cases were studied further with two conjectures being made. The first one involved polynomial variations: the branching points clustered around the unit circle as the degree $n$ approaches infinity. The second one involved factorial variations $v_{j,k} = \frac{n!}{jk!(n-j-k)!}$ and $v_{j,k} = \frac{n!}{(j+k)!(n-j-k)!}$: the radii of branching points approach a log-normal distribution, with the first having a median of 1, whereas the second
has an increasing median as $n \to \infty$. The log distributions also have increasing $\sigma$ as $n \to \inf$. Furthermore, we explored the relative radial distribution of monodromy group elements of bivariate polynomials. Our preliminary study and results showed that this subject is rich and promising. More theoretical analysis may be investigated to support our conjectures, and more interesting numerical phenomena about the distribution of monodromy group transpositions may be observed and explained in the future.

References

