Random Matrix Pencils, Branching Points, and Monodromy

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1 Introduction

We define complex projective n-space $\mathbb{CP}^n$ as $\mathbb{C}^{n+1} \setminus \{0\}$ under the equivalence relation $(z_0 : z_1 : ... : z_n) \sim (cz_0 : cz_1 : ... : cz_n)$ where $c \in \mathbb{C} \setminus \{0\}$. A complex projective curve is then the set of points $(a_0 : a_1 : ... : a_n) \in \mathbb{CP}^n$ such that $p(a_0, a_1, ..., a_n) = 0$ for a fixed complex homogeneous polynomial $p(z_0, z_1, ..., z_n)$. Such a curve is a 2-dimensional, real, orientable Riemannian manifold, or Riemann surface.

For a non-constant holomorphic map $f : X \to Y$ between compact, connected Riemann surfaces X and Y, f is locally a covering map. The preimage $f^{-1}(p)$ of a point $p \in Y$ generally consists of d distinct points, where d is the degree of the map f. If the preimage $f^{-1}(p)$ consists of fewer than d distinct points, p is called a branching point of the map f. In appropriate local coordinates at each point $x \in f^{-1}(p)$, the map f looks like $z \mapsto z^k$ for some positive integer $k \in \mathbb{Z}^+$ called the ramification index of x. A branching point p is called simple if some point $x \in f^{-1}(p)$ has ramification index 2 and the rest have index 1. Let $br(f)$ be the set of branching points of f and fix a base point $y \in Y$ which is not a branching point. Each loop in $Y \setminus br(f)$ with base point y lifts to d distinct paths in X with endpoints in $f^{-1}(y)$, and thus we get a group homomorphism from the fundamental group $\pi_1(Y \setminus br(f))$ to the symmetric group $S_{f^{-1}(y)} \simeq S_d$ called the monodromy. The image of a simple branching point is a transposition, and the image of $br(f)$ is called the monodromy group.

For complex $n \times n$ matrices A and B, define the characteristic polynomial $\chi(t, \lambda) = \det(A + tB + \lambda I)$. By considering the homogenized polynomial $\tilde{\chi}(t, \lambda, z) = z^n \chi(t/z, \lambda/z)$, let R be the zero set of χ in $\mathbb{CP}^2$. Let $f : R \to \mathbb{CP}^1$ be the projection onto the t-coordinate. The branching points of this map are precisely the values of t such that $A + tB$ has repeated eigenvalues. In this project, we studied the distribution of the branching points of this map where A and B are random matrices sampled from the Gaussian Orthogonal Ensemble GOE_n or from the Gaussian Unitary Ensemble GUE_n and investigate the monodromy of each.

2 Distributions of Branching Points

We consider two probability spaces of random matrices, the Gaussian Orthogonal Ensemble GOE_n and the Gaussian Unitary Ensemble GUE_n, so named because their distributions are invariant under conjugation by orthogonal and unitary matrices, respectively. The ensemble GOE_n consists of real symmetric matrices $X = (X_{ij})_{i,j=1}^n$ where the diagonal elements X_{ii}, $1 \le i \le n$ are independently sampled from the normal distribution $\sqrt{2}N(0,1)$ and the above-diagonal elements X_{ij}, $1 \le i < j \le n$ are independently sampled from the standard normal distribution $N(0,1)$. Similarly, GUE_n consists of complex Hermitian matrices $X = (X_{ij})_{i,j=1}^n$ where the diagonal elements X_{ii}, $1 \le i \le n$ are independently sampled from the standard normal distribution $N(0,1)$ and the real and imaginary parts of the above-diagonal elements X_{ij}, $1 \le i < j \le n$ are independently sampled from the normal distribution $\frac{1}{\sqrt{2}}N(0,1)$. In this section, we first calculate the distributions of the branching points for random matrix pencils of GUE_2 and GOE_2 matrices. Then, we present experimental results for the distributions of branching points for these ensembles for larger values of n, Finally, we explore symmetries of the distributions of the branching points for GOE_n and GUE_n which restrict the possible theoretical distributions for all n.

The ensemble GUE_n is invariant under conjugation by unitary matrices, so for GUE_2 matrices A and B, we can conjugate $A + tB$ with a unitary matrix such that A is diagonal. Therefore, we will assume that A is a diagonal matrix, i.e., $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. Notice that since A is diagonal, λ_1 and λ_2 are the eigenvalues of A. Without a loss of generality, $\lambda_1 < \lambda_2$. Then, we can shift our matrix pencil so that $A = \begin{pmatrix} 0 & 0 \\ 0 & \Delta \end{pmatrix}$, where $\Delta = \lambda_2 - \lambda_1$. We know that the branching points of our projection are exactly the zeroes of the discriminant of our characteristic polynomial $\chi(\lambda, t) = \det(A + tB + \lambda I) = t^2((b_{22} - b_{11})^2 + 4|b_{12}|^2) + t2\Delta(b_{22} - b_{11}) + \Delta^2$. Since this is a real polynomial, our branching points come in complex conjugate pairs $(\tau, \bar{\tau})$ where

$$\tau = \frac{-(b_{22} - b_{11})\Delta + 2i|b_{12}|\Delta}{(b_{22} - b_{11})^2 + 4|b_{12}|^2}.$$

In order to find the distribution of τ, we will first find the conditional distribution of τ, assuming a constant value for Δ. Set $X = \frac{-(b_{22}-b_{11})}{\Delta}$ and $Y = \frac{2|b_{12}|}{\Delta}$ so that $\tau = \frac{1}{X - Yi}$. Since $b_{11}, b_{22} \sim N(0,1)$, then $b_{22} - b_{11} \sim N(0,2)$, and hence, $X \sim N(0, \frac{2}{\Delta^2})$. Therefore, the conditional PDF of X is

$$\mathcal{P}_\Delta(x) = \frac{1}{\frac{\sqrt{2}}{\Delta}\sqrt{2}\sqrt{\pi}} e^{-x^2\Delta^2/4} = \frac{\Delta}{2\sqrt{\pi}} e^{-x^2\Delta^2/4}.$$

Since $\mathrm{Re}(b_{12}), \mathrm{Im}(b_{12}) \sim N(0, \frac{1}{2})$, then $\frac{1}{1/\sqrt{2}}|b_{12}| \sim \chi_2$. Thus, the conditional PDF of Y is

$$\mathcal{P}_\Delta(y) = \frac{\Delta^2 y}{2} e^{-y^2\Delta^2/4}.$$

Taking the joint distribution of X and Y gives us the conditional distribution of $\frac{1}{\tau}$. Since b_{12} is independent from b_{11} and b_{22}, then X and Y are also independent, so the conditional PDF of $\frac{1}{\tau}$ is the product of the PDFs of X and Y,

$$\mathcal{P}_\Delta(x,y) = \mathcal{P}_\Delta(x)\mathcal{P}_\Delta(y) = \frac{\Delta^3}{4\sqrt{\pi}} y e^{-(x^2+y^2)\Delta^2/4}.$$

To find the conditional distribution for τ, we will change variables in order to write τ in terms of its real and imaginary components. Set $U = \frac{X}{X^2+Y^2}$ and $V = \frac{Y}{X^2+Y^2}$ so that $\tau = \frac{1}{X-Yi} = U + Vi$. Then the Jacobian is $\left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \frac{1}{(x^2+y^2)^2} = (u^2+v^2)^2$, so the conditional distribution for τ in terms of U and V is

$$\mathcal{P}_\Delta(u,v) = \frac{\mathcal{P}_\Delta(x,y)}{\left|\frac{\partial(u,v)}{\partial(x,y)}\right|} = \frac{v\Delta^3 e^{-\Delta^2/4(u^2+v^2)}}{4\sqrt{\pi}(u^2+v^2)^3}.$$

As $U = \mathrm{Re}(\tau)$ and $V = \mathrm{Im}(\tau)$, the conditional distribution of τ for a given value of Δ is

$$\mathcal{P}_\Delta(\tau) = \frac{\mathrm{Im}(\tau)\Delta^3 e^{-\Delta^2/4|\tau|^2}}{4\sqrt{\pi}|\tau|^6}.$$

So, in order to find the distribution of τ, we need the joint distribution for the eigenvalues λ_1 and λ_2 of matrices belonging to GUE_2,

$$\mathcal{P}(\lambda_1,\lambda_2) = \frac{1}{2\pi}(\lambda_2-\lambda_1)^2 e^{-(\lambda_1^2+\lambda_2^2)/2}.$$

Therefore, the distribution for τ is

$$\begin{aligned}\mathcal{P}(\tau) &= \int_{-\infty}^{\infty}\int_{\lambda_1}^{\infty} \mathcal{P}(\lambda_1,\lambda_2)\mathcal{P}_\Delta(\tau,\bar\tau)\, d\lambda_2\, d\lambda_1 \\ &= \int_{-\infty}^{\infty}\int_{\lambda_1}^{\infty} \frac{\Delta^3}{8\pi^{3/2}}(\lambda_2-\lambda_1)^2 \frac{\mathrm{Im}(\tau)}{|\tau|^6} e^{\frac{-\Delta^2}{4|\tau|^2} - \frac{(\lambda_1^2+\lambda_2^2)}{2}}\, d\lambda_2\, d\lambda_1 \\ &= \frac{8\mathrm{Im}(\tau)}{\pi(1+|\tau|^2)^3}.\end{aligned}$$

However, since we assumed that $\mathrm{Im}(\tau) > 0$, the distribution for τ over the whole plane is

$$\mathcal{P}(\tau) = \frac{4|\mathrm{Im}(\tau)|}{\pi(1+|\tau|^2)^3}.$$

Using the same methods as above, we calculated the distribution of the branching points of GOE_2 matrices. As in the previous case, $\tau = \frac{-(b_{22}-b_{11})\Delta + 2i|b_{12}|\Delta}{(b_{22}-b_{11})^2+4|b_{12}|^2} = \frac{1}{X-Yi}$ where $X = \frac{b_{11}-b_{22}}{\Delta}$ and $Y = \frac{2|b_{12}|}{\Delta}$. Note that $X \sim N(0, \frac{4}{\Delta^2})$ and $Y \sim \frac{2}{\Delta}\chi_1$. Thus, $\mathcal{P}_\Delta(x) = \frac{\Delta}{2\sqrt{2\pi}} e^{-\frac{x^2\Delta^2}{8}}$ and $\mathcal{P}_\Delta(y) = \frac{\Delta}{\sqrt{2\pi}} e^{-\frac{y^2\Delta^2}{8}}$. Therefore, $\mathcal{P}_\Delta(x,y) = \mathcal{P}_\Delta(x)\mathcal{P}_\Delta(y) = \frac{\Delta^2}{4\pi} e^{-\frac{\Delta^2(x^2+y^2)}{8}}$.
Letting $U = \frac{X}{X^2+Y^2}$ and $V = \frac{Y}{X^2+Y^2}$ so that $\tau = \frac{1}{X-iY} = U + iV$, we get

$$\mathcal{P}_\Delta(u,v) = \frac{\Delta^2}{4\pi(u^2+v^2)^2} e^{-\frac{\Delta^2}{(u^2+v^2)8}}.$$

Therefore,

$$\mathcal{P}_\Delta(\tau) = \frac{\Delta^2}{4\pi \mid \tau \mid^4} e^{-\frac{\Delta^2}{8|\tau|^2}}.$$

The distribution of the eigenvalues of a GOE_2 matrix is

$$\mathcal{P}(\lambda_1, \lambda_2) = \frac{1}{4\sqrt{2\pi}} e^{-\dfrac{\lambda_1^2 + \lambda_2^2}{4}} (\lambda_2 - \lambda_1),$$

where $\lambda_1 \leq \lambda_2$.
Thus, the distribution of τ is

$$\begin{aligned}\mathcal{P}(\tau) &= \int_{-\infty}^{\infty} \int_{\lambda_1}^{\infty} \frac{1}{4\sqrt{2\pi}} e^{-\dfrac{\lambda_1^2 + \lambda_2^2}{4}} (\lambda_2 - \lambda_1) \frac{\Delta^2}{4\pi \mid \tau \mid^4} e^{-\frac{\Delta^2}{8|\tau|^2}} \, d\lambda_2 \, d\lambda_1 \\ &= \frac{2}{\pi(1+ \mid \tau \mid^2)^2}\end{aligned}$$

Since we were assuming that the imaginary part of τ is positive, to get the real PDF of τ we must divide by 2, so

$$\mathcal{P}(\tau) = \frac{1}{\pi(1+ \mid \tau \mid^2)^2}.$$

These theoretical results for the distributions of branching points for GUE_2 and GOE_2 are further supported by the experimental results which we gathered, shown in Figure 1.

Notice that the distribution for the branching points of GOE_2 is clearly the uniform distribution on the sphere. This realization led us to observe that SO_2 acts on $D_n \subset H_n \times H_n \times \mathbb{C}P^1$ the ste of triples $A, B, [u : v]$ such that $[u : v]$ is a branch point of the pair A, B. SO_2 acts on $H_n \times H_n$ by

$$\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \alpha A + \beta B \\ -\beta A + \alpha B \end{pmatrix}$$

and on $\mathbb{C}P^1$ by

$$\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} [u : v] = [\alpha u + \beta v : -\beta u + \alpha v]$$

Observe that if $\phi \in [0, 2\pi]$ and $\alpha = \cos(\phi), \beta = \sin(\phi)$, then $[\alpha u + \beta v : -\beta u + \alpha v]$ is a branch point of the pair $\alpha A + \beta B, -\beta A + \alpha B$. Indeed,

$$\begin{aligned}&(\alpha u + \beta v)(\alpha A + \beta B) + (-\beta u + \alpha v)(-\beta A + \alpha B) \\ &= \alpha^2 uA + \alpha\beta vA + \alpha\beta uB + \beta^2 B + \beta^2 uA - \alpha\beta uB - \alpha\beta vA + \alpha^2 vB \\ &= uA + vB.\end{aligned}$$

Hence SO_2 acts on D_n, and this action commutes with the projections $\pi : D_n \to H_n \times H_n$ and $\zeta : D_n \to \mathbb{C}P^1$. Moreover, the action on $H_n \times H_n$ preserves the densities of the GOE_n and GUE_n ensembles. These densities are given by $C_{GOE_n} e^{-\frac{n}{4}\mathrm{tr}(M^2)}$ and $C_{GUE_n} e^{-\frac{n}{2}\mathrm{tr}(M^2)}$ respectively, so the density of the pair A, B is determined by $\mathrm{tr}(A^2 + B^2)$ and

$$\begin{aligned}&\mathrm{tr}((\alpha A + \beta B)^2 + (-\beta A + \alpha B)^2) \\ &= \mathrm{tr}(\alpha^2 A^2 + \alpha\beta AB + \alpha\beta BA + \beta^2 B^2 + \beta^2 A^2 - \alpha\beta AB - \alpha\beta BA + \alpha^2 B) \\ &= \mathrm{tr}(A^2 + B^2).\end{aligned}$$

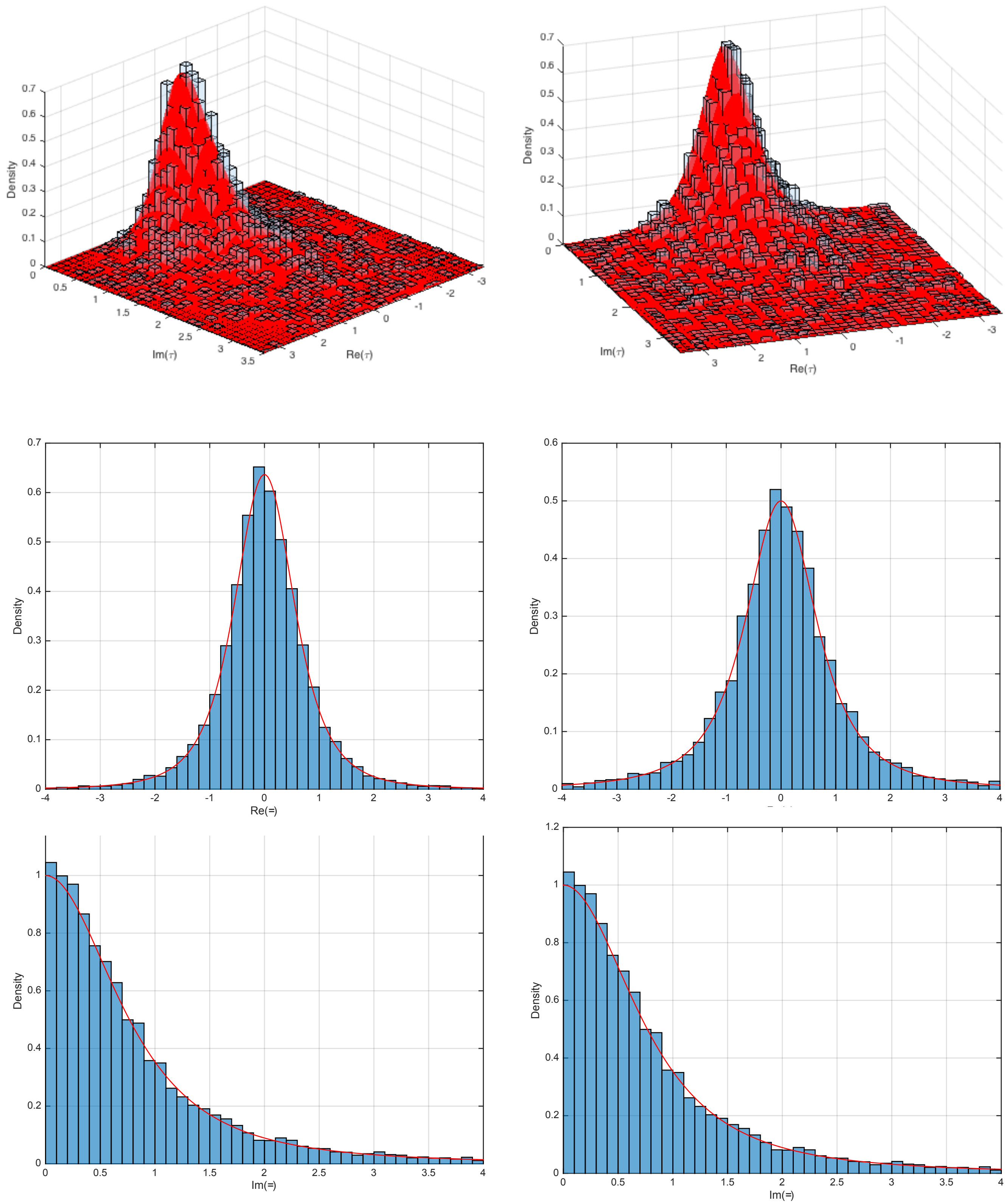

Figure 1: The density of branching points for GUE_2 on the left and GOE_2 on the right.

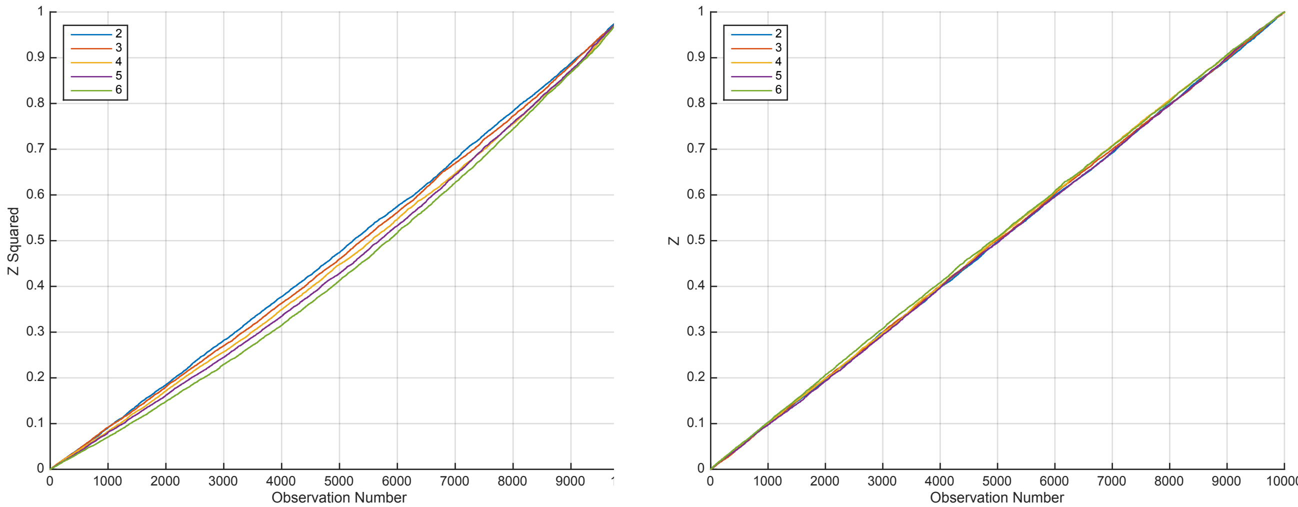

Figure 2: The ordered values of z^2 of the branching points of GUE_n on the left and the ordered values of z of the branching points of GOE_n matrices on the right, for $n \in \{2, \ldots, 6\}$.

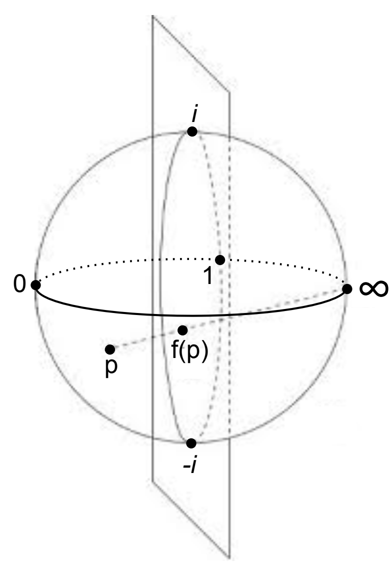

Figure 3: The circle action on GUE_n and GOE_n.

$$\begin{array}{ccc} H_n \times H_n \times \mathbb{C}P^1 & \xrightarrow{\zeta} & \mathbb{C}P^1 \\ \big\downarrow{\scriptstyle \pi} & & \\ H_n \times H_n & & \end{array}$$

The density of the branching points in $\mathbb{C}P^1$ is given by the pushforward of the pullback of the density on $H_n \times H_n$. That is, the measure $\tilde{\mu}$ of a set $E \subset \mathbb{C}P^1$ is given by $\mu(\pi(\zeta^{-1}(E))$. $\tilde{\mu}(g \cdot E) = \mu(\pi(\zeta^{-1}(g \cdot E))) = \mu(\pi(g \cdot \zeta^{-1}(E))) = \mu(g \cdot \pi(\zeta^{-1}(E))) = \mu(\pi(\zeta^{-1}(E))) = \tilde{\mu}(E)$, so we conclude that the densty of the branching points is invariant under this action by SO_2. Changing to cylindrical coordinates on $\mathbb{C}P^1$,

$$\phi(A, B, (\theta, z)) = (\alpha A + \beta B, -\beta A + \alpha B, (\theta - 2\phi, z))$$

so the action is a rotation of the sphere about the imaginary axis. Applying this change of coordinates to our distribution of the branching points of GUE_2, we get the distribution $\mathcal{P}(\theta, z) = \frac{|z|}{\pi}$, where $0 \leq \theta \leq 2\pi$ and $0 \leq z \leq 1$. Integrating over θ gives the marginal PDF $\mathcal{P}(z) = 2|z|$, therefore the CDF for z is z^2. We then found 10,000 branching points for GUE_n matrices for each $n \in \{2, \ldots, 6\}$ and plotted their corresponding z^2 values in increasing order, shown in Figure 2. In the same way, our change of coordinates for the distribution of the branching points of GOE_2 yields the distribution $\mathcal{P}(\theta, z) = \frac{1}{2\pi}$, where $0 \leq \theta \leq 2\pi$ and $0 \leq z \leq 1$. Then, the marginal PDF of z is $\mathcal{P}(z) = 1$, and integrating gives that the CDF for z is z. Plotting the z values of 10,000 branching points for GOE_n matrices for each $n \in \{2, \ldots, 6\}$, as in Figure 2, suggests that the branching points of GOE_n are uniformly distributed on the sphere for all n.

3 Monodromy

For general A and B in GOE_n, the branching points are all simple and come in complex conjugate pairs, $\binom{n}{2}$ of which lie in the upper half plane. Let $\tau_1, \tau_2, ..., \tau_{\binom{n}{2}}$ be the branching points in the upper half plane ordered by real part. Since the branching points are simple, let $\sigma_1, \sigma_2, ..., \sigma_{\binom{n}{2}}$ be the associated transpositions in the monodromy group. These transpositions

(i) generate the symmetric group S_n and

(ii) their product $\sigma_1 \sigma_2 ... \sigma_{\binom{n}{2}}$ is $\begin{pmatrix} 1 & 2 & ... & n \\ n & n-1 & ... & 1 \end{pmatrix}$

For $n = 3$, it is easy to see that there are only 8 words of $\binom{n}{2}$ transpositions which satisfy (i) and (ii). For $n = 4$, we wrote a program to loop through all $\binom{4}{2}^{\binom{4}{2}} = 46,656$ such words and checked properties (i) and (ii). There are 3872 words that satisfy (ii), which is easy to verify. For S_4, property (i) is equivalent to the word containing at least 3 distinct transpositions and the transpositions $\sigma_1, \sigma_2, ..., \sigma_{\binom{4}{2}}$ having no common fixed points. This check eliminates 32 words, for a total of 3840 possible words. Next, we experimentally determined the frequency of the words that arise from the monodromy for matrices sampled from GOE_3, but we lacked the computational resources for $n = 4$.

We wrote a short MATLAB program to compute the transposition associated to a branching point τ of a pair of matrices (A, B). The program calculates the eigenvalues of $A + (\mathrm{Re}(\tau) + \epsilon \mathrm{Im}(\tau))B$ as ϵ runs from 0 to 1 in steps of 0.01. A typical plot of the eigenvalues during this process is shown in Figure 4. At $\epsilon = 0$ all of the eigenvalues are real, so we number them in increasing order. For each new ϵ, the new eigenvalues are assigned the same numbers as the closest eigenvalues from the previous ϵ. Then, when two eigenvalues collide at $\epsilon = 1$, the numbers assigned to the two eigenvalues give the transposition for τ. By following this procedure for each of the branching points in the upper half plane, in order of increasing real part, as shown in Figure 5, we obtain the monodromy associated to (A, B). Because errors can occur if the real parts of different branching points are very similar, we rejected matrices with this property when gathering monodromy statistics. The resulting statistics for GUE_3 and GOE_3 matrices are shown in Figure 6 and 7, respectively.

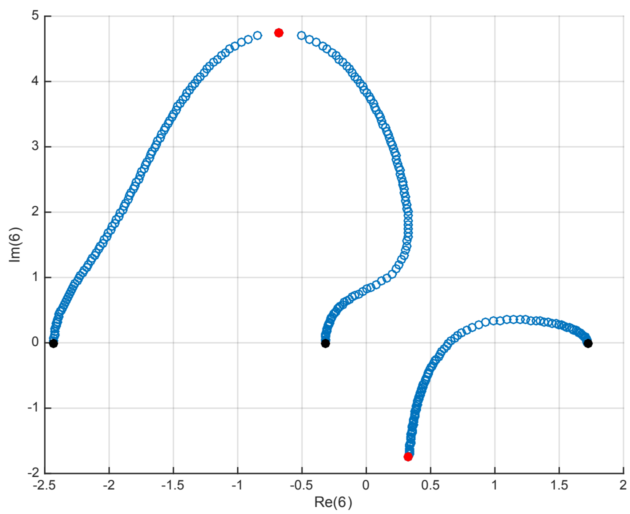

Figure 4: The first and second eigenvalues collide as we approach the branching point, giving the transposition (12).

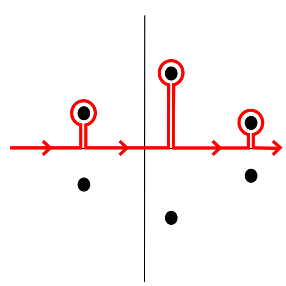

Figure 5: Path which gives the transpostion sequence.

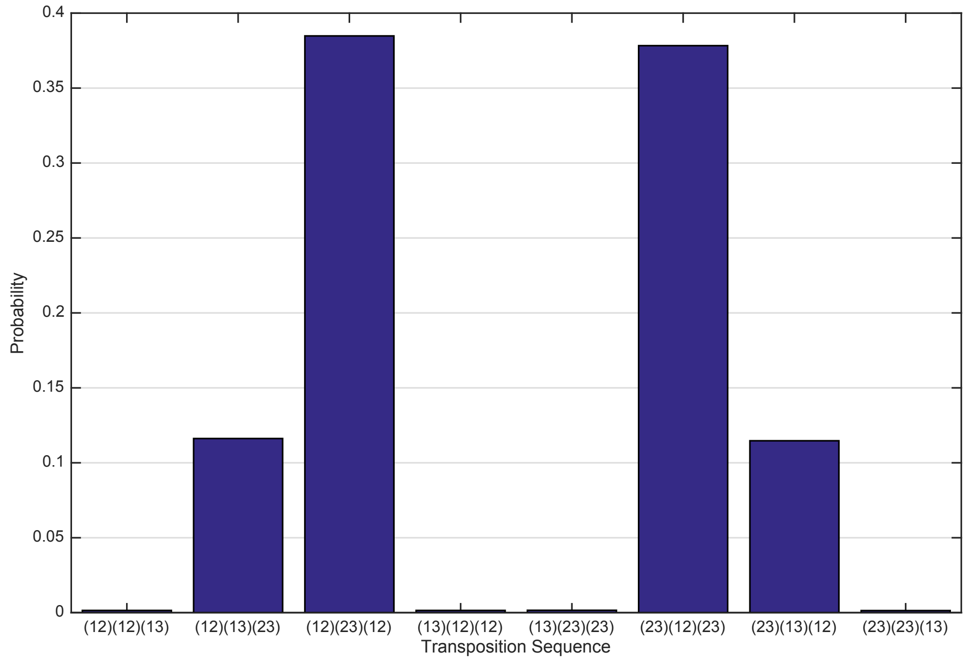

Figure 6: The statistics for the monodromy of GUE_3 matrices.

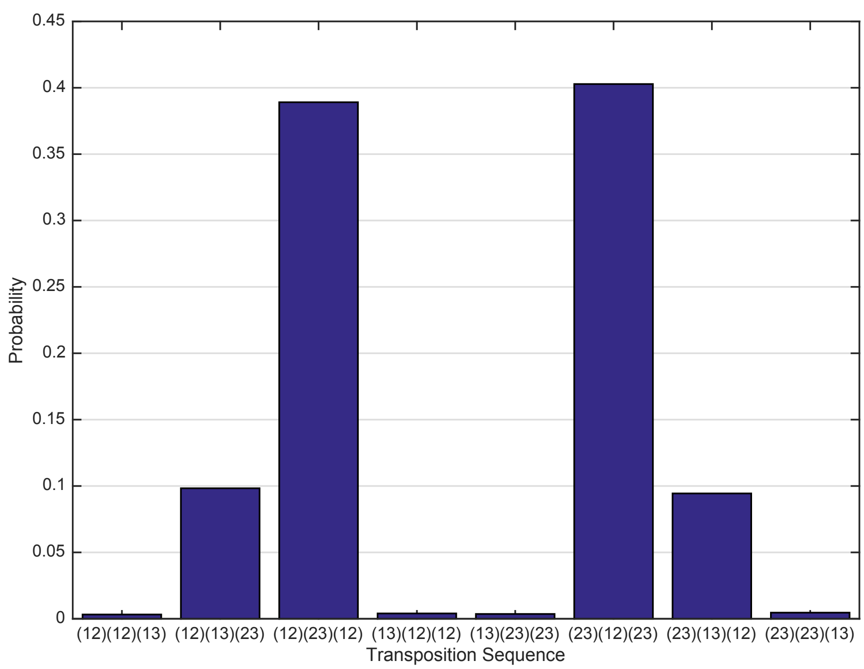

Figure 7: The statistics for the monodromy of GOE_3 matrices.

Notice that the statistics for the monodromy of GUE_n and GOE_n are invariant under conjugation by the inverse transposition $\begin{pmatrix} 1 & 2 & ... & n \\ n & n-1 & ... & 1 \end{pmatrix}$ as well as under reversing the order of the transpositions. These symmetries can be explained as consequences of the symmetries of the ensembles.

If the matrix $A+tB$ has eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$, then the matrix $-A-tB$ has eigenvalues $-\lambda_1, -\lambda_2, ... - \lambda_n$. These matrix pencils share the same branching points, and if a loop in $\mathbb{C}P^1$ permutes the eigenvalues of $A + tB$, then it must apply the same permutation to the eigenvalues of $-A - tB$. However, when we compute the monodromy associated to a pair of matrices in our ensembles, we order the (real) eigenvalues for real t, and the transpositions associated to each branching point are written with respect to this ordering. Since the eigenvalues of $-A - tB$ will have the opposite ordering as those of $A + tB$, the monodromy associated to the pair $(-A, -B)$ will be the mondodromy of (A, B), conjugated by $\begin{pmatrix} 1 & 2 & ... & n \\ n & n-1 & ... & 1 \end{pmatrix}$. Since the pairs (A, B) and $(-A, -B)$ have the same density, each monodromy will appear with the same frequency as its conjugate.

The other symmetry of the data, invariance under reversing the order of the transpositions, can be similarly explained by the equal density of the pairs (A, B) and $(A, -B)$. If the branching points of $A + tB$ are $\tau_1, \tau_2, ...\tau_m$, then the branching points of $A - tB$ are $-\tau_1, -\tau_2, ... - \tau_m$. Branching points come in conjugate pairs, and the same transpositions are associated to these pairs, so if $\tau_1, \tau_2, ...\tau_n$ are the branching points of $A + tB$ in the upper half plane, $\overline{-\tau_1}, \overline{-\tau_2}, ...\overline{-\tau_n}$ are the branching points of $A - tB$ in the upper half plane. Since we order these by their real parts, which have been inverted, it now remains to show that the transposition associated to (A, B, τ_i) is the same as that associated to $(A, -B, \overline{-\tau_i})$. Since the transposition associated to a branching point is the same as that associated to its conjugate, we instead consider $(A, -B, -\tau_i)$.

Observe that the transposition associated to (A, B, τ_i) is determined by the eigenvalues of

$$A + (\mathrm{Re}(\tau_i) + \epsilon \mathrm{Im}(\tau_i))B$$

for $0 \leq \epsilon \leq 1$, and in the same way the transposition asociated to $(A, -B, -\tau_i)$ is determined by

$$A + (\mathrm{Re}(-\tau_i) + \epsilon \mathrm{Im}(-\tau_i))(-B) = A + (\mathrm{Re}(\tau_i) + \epsilon \mathrm{Im}(\tau_i))B$$

These coincide, and we conclude that the monodromy associated to $(A, -B)$ is the reverse of that associated to (A, B).