LINEAR SEARCH ON TWO RAYS

ARGEN WEST

Abstract. To investigate the properties of an optimal trajectory of a linear search problem on two rays with a symmetric exponential distribution.

1. Introduction

The goal for this project is to investigate the linear search problem on two rays. Suppose an object is hidden somewhere on two rays with a given probability distribution, f(x). A searcher follows a search plan in hopes of finding the hidden object. The search plan is a sequence of excursions which tells the searcher how far out to go on a ray before returning to the beginning. For each excursion on a ray the searcher collects everything and returns to the beginning. The searcher can only verify if the hidden object was found or not by bringing it to the origin for verification. If it is determined that the object was not found on the ray the searcher then does an excursion on the other ray. The searcher will repeat this process until the hidden object is found.

Search Plan and Expected Cost. A search plan, $\mathbf{x} = \{x_i\}_0^\infty$ where $x_i \geq 0$, $x_0 = 0$ and $\lim_{i\to\infty} x_i = \infty$, is a sequence of turning points on which a the searcher turns around and returns to the beginning. The turning points corresponding to turns on the first ray are x_{2n-1} and turning points corresponding to turns on the second ray are x_{2n} for $n \geq 1$. Each excursion adds the length of the excursion times the probability of not finding the object on the previous excursions on each ray to the expected cost. Hence, the expected cost function on two rays associated with a set of turning points is given by,

$$C(\mathbf{x}) = x_1 + \sum_{i=2}^{\infty} x_i(f(x_{i-1}) + f(x_{i-2}))$$

Questions and Goals. A few questions we are interested in for the search problem on two rays are,

- Does an optimal sequence exist?
- Is the optimal sequence monotone increasing?

Plugging in $\mathbf{x} = \{2^k\}_0^\infty$ to the expected cost function yields a converging sum. This may make one tempted to say an optimal sequence exist. However, it only shows the set $\{C(\mathbf{x})|\mathbf{x} = \{x_i\}_0^\infty, x_i \geq 0, x_0 = 0, \lim_{i\to\infty} x_i = \infty\}$ must have a finite infimum. Also, one might think it should be monotone increasing because why would a searcher search a shorter distance than it just traveled. In the one ray case it would be hard to argue with that logic. For example in the one ray case if the searcher goes 10 units out then 8 units out, it has only covered 10 units of the ray. However, in the two ray case if the searcher went out 10 units and then 8 units, it would have covered 18 units between both rays. Hence, it must be proved that an optimal sequence exists and that it is monotone increasing for the two ray case.

2. Exponential Distribution

Before we try to investigate linear search problem on two rays with an arbitrary distribution we will try to gain some intuition. For that reason we will look the linear search problem on two rays with the exponential distribution, $f(x) = \frac{e^{-x}}{2}$. Note that this distribution is symmetric, i.e. the same on each ray.

Findings. We proved that if an optimal sequence exist, it must be monotone. Proving that monotonicity holds wasn't as straight forward as one would hope. First we had to show that an optimal search plan could not have an excursion that was greater than both the next two excursions or less than both the previous two excursions. Which yielded the following two propositions:

Proposition 2.1. *A sequence* $\boldsymbol{x}$ *with* $x_n > x_{n+1}$ *and* $x_n > x_{n+2}$ *for some* n *is not optimal.*

Proof. Suppose $x_n > x_{n+1}$ and $x_n > x_{n+2}$. Remove x_n from $\mathbf{x}$ and call the new sequence $\mathbf{x^*}$. Then it can be shown $C(\mathbf{x}) - C(\mathbf{x}^*) > 0$ □

Proposition 2.2. *A sequence* $\boldsymbol{x}$ *with* $x_n < x_{n-1}$ *and* $x_n < x_{n-2}$ *for some* n *is not optimal.*

Proof. Suppose $x_n < x_{n-1}$ and $x_n < x_{n-2}$. Remove x_n from $\mathbf{x}$ and call the new sequence $\mathbf{x^*}$. Then it can be shown $C(\mathbf{x}) - C(\mathbf{x}^*) > 0$ □

We then used the above propositions to prove that every time the searcher has an excurison on a ray, it must be greater than the previous excursion on that same ray. Intuitively this makes sense. It would not make sense for a searcher have to have had covered less units on a ray than it did last time it was on that same ray.

Theorem. *A sequence* $\boldsymbol{x}$ *that is optimal will have* $x_n < x_{n+2}$.

Proof. Let $\mathbf{x}$ be optimal. By Proposition 2.1 $x_n < x_{n+1}$ or $x_n < x_{n+2}$. If $x_n < x_{n+2}$, then we are done. Suppose $x_n > x_{n+2}$. Thus, $x_n < x_{n+1}$. Hence, $x_{n+2} < x_n < x_{n+1}$. However, $x_{n+2} < x_n$ and $x_{n+2} < x_{n+1}$ violates Proposition 2.2. So, we must have $x_n < x_{n+2}$. □

Finally using the propositions and theorem above we were able to show monotonicity holds for an optimal sequence.

Theorem. *A sequence* $\boldsymbol{x}$ *with* $x_n < x_{n-1}$ *is not optimal.*

Proof. Suppose $x_n < x_{n-1}$. Switch x_n and x_{n-1} in $\mathbf{x}$ and call the new sequence $\mathbf{x^*}$. i.e. $\mathbf{x}$=$x_0, x_1, ..., x_{n-1}, x_n, x_{n+1}, ...$ and $\mathbf{x^*}$=$x_0, x_1, ..., x_n, x_{n-1}, x_{n+1}, ...$. Then it can be shown $C(\mathbf{x}) - C(\mathbf{x}^*) \geq 0$ □

Hence, an optimal sequence must be monotone.

Currently Investigating. Although an optimal sequence must be monotone increasing it must still be shown that such a sequence exists. So far we have shown, an optimal sequence will have no accumulation points near 0 or any other number. In other words if we can find a sequence of sequences $\{\mathbf{x}_i\}$ converging to an optimal sequence, it will not be the degenerate $0, 0, 0, ...$ sequence. Thus we are looking to show that there is a sequence of sequences ,$\{\mathbf{x}_i\}$, converging to an optimal sequence $\mathbf{x}^*$ such that $C(\mathbf{x}^*)$ is the infimum of $\{C(\mathbf{x}) | \mathbf{x} = \{x_i\}_0^\infty, x_i \geq 0, x_0 = 0, \lim_{i\to\infty} x_i = \infty\}$. Hence, $C(\mathbf{x}^*)$ would be the optimal cost with the optimal sequence $\mathbf{x}^*$.

3. Future Questions

After gaining intuition from the exponential case we hope to extended the findings to more general statements and look at other questions. A few that might be interesting to investigate are:

- What happens which each ray has a different distribution function?
- What happens if the distribution is not lipschitz?
- What happens if there are n rays?
- Can we do numerics estimate the cost function?
- Would an optimal start at the separatrix?

4. Acknowledgements

The author would like to thank Professor Vadim Zharnitsky for his guidance in this project. The author acknowledges support from National Science Foundation grant DMS 1345032 "MCTP: PI4: Program for Interdisciplinary and Industrial Internships at Illinois."