The Oscillators of Synchronization

Timothy Ferguson, Alonza Terry, Mary Angelica Tursi

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Introduction

The analysis of the heart is useful as its ion concentrations (growth and decay) oscillate at various frequencies while all are synchonized. The aim of this paper is to access the importance of the synchronization process and its applications to mathematics. The common model is the (*Kuramoto*) model which is a set of ODE's:

$$\frac{\partial \theta_i}{\partial t} = \omega_{it} \sum_j \gamma_{ij} sin(\theta_i - \theta_j).$$

We note that θ_i = phase of i^{th} oscillator and ω_i = the natural frequency of the i^{th} oscillator. Another important aspect of this project is to consider the eigenvalues of the Jacobian since they are vital.

$$\begin{cases} Re(\lambda_i) > 0. & \text{UNSTABLE} \quad \text{(exponential growth)} \\ Re(\lambda_i) < 0. & \text{STABLE} \quad \text{(exponential decay)} \end{cases}$$

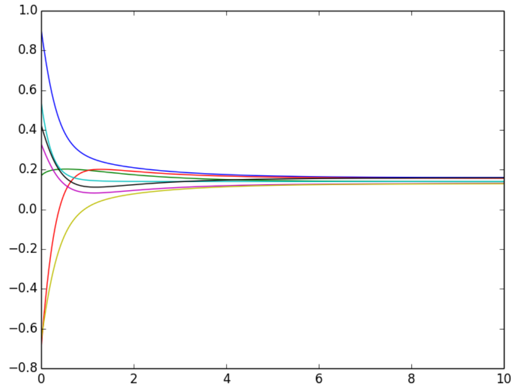
For the Kuramoto Model, the fixed points are:

$$\omega_i + \gamma \sum_j sin(\theta_j - \theta i)\gamma_{ij} = 0.$$

The first part of the project was to implement an ODE on some graph and determine a steady state. In addition, we had to find a set of frequencies, determine the Jacobian and find several periodic functions that will lead to synchronization and some with unique steady states.

Determining A Steady State

We implemented an ODE on a given graph and found a steady state which synchronized.



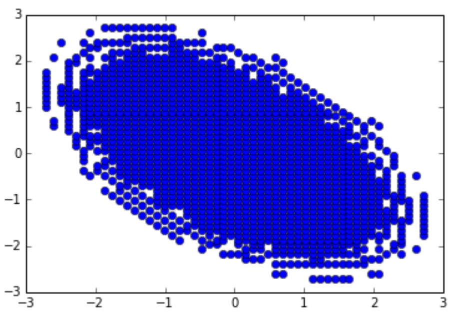
(a) Steady State

Set of Frequencies

When the complete graph K_3 is considered, a certain set of frequencies results in synchronization. The elements of the set are determined by two of the three frequencies, and these two frequencies form a coordinate which is an element of the set. To determine whether or not the nodes synchronize, the norm of the difference between the last two values of each θ is computed. If the norm is less than some epsilon (here epsilon was 0.1), the graph most likely synchronized. The example of frequency sets is shown below. When

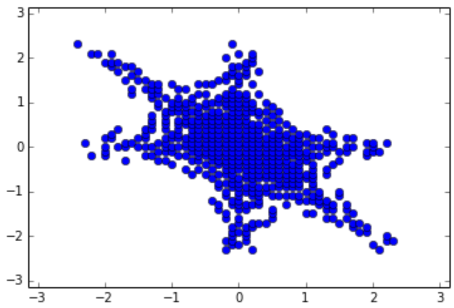
$$\frac{\partial \theta_i}{\partial t} = \omega_{it} \sum_j \gamma_{ij} sin(\theta_i - \theta_j).$$

The set of weights which result in a synchronization is in the form of an ellipse.



(a) Elipse

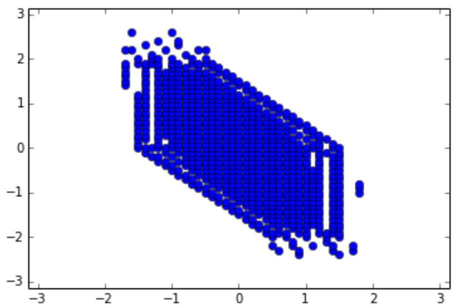
If the weights are multiplied by a certain factor, the ellipse is stretched by that factor. However, if multiplied by -1, the result is a hexagonal figure with intentations protruding into the original ellipse:



(a) Frequency set for the negative sine

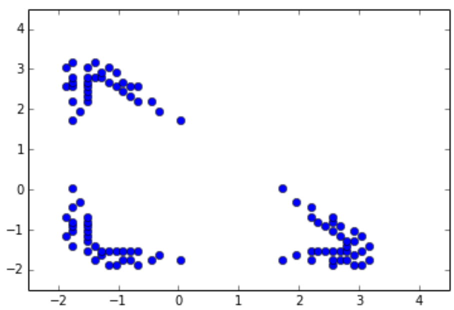
This figure, like that of the ellipse, can also be extended if multiplied by a certain constant.

The path of length 2 also has a similar frequency set in terms of its range when synchronized according to the original sine function, but it is far more hexagonal in shape:



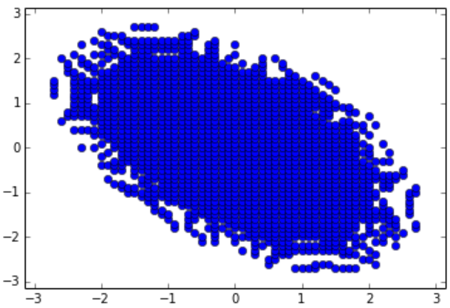
(a) Frequency set for sine in a path

Attempts were made to see if the ellipse could be skewed in other directions. Adding constants sharply shrinks the frequency set. However, adding the cosine function yielded a triangluar result, even though the leftward skew remains:

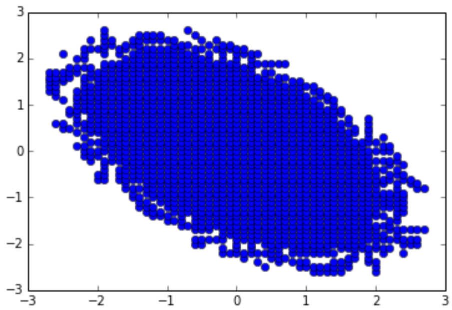


(a) Frequency set for the sine and cosine

When the angles are multiplied by a constant, the plot of the set retains some of its structure, but over time becomes more diluted around the edges. The following plot below shows the frequency set for sin(2x) and sin(3x) respectively. On the other hand, parts of the graph which retain a solid contour remain the same size and shape. Since we are using numerical methods, it is possible for some points to make it into the frequency plot without necessarily synchronizing. Yet the fact that the edges begin to dilute while retaining the original solid ellipse suggests that intuitively, when chance for error is removed, the frequency set for any sin(nx) function remains the same.



(a) Frequency set for the elipse sin(2x)



(a) Frequency set for the elipse sin(3x)

The Jacobian

We then investigated the eigenvalues of the Jacobian J of the vector field of the Kuramoto system. Let

$$f_i(\theta) = \omega_i + \sum_j \gamma_{ij} \sin(\theta_j - \theta_i),$$

then

$$J_{ij} = \frac{\partial}{\partial \theta_j} f_i(\theta) = -\sum_j \gamma_{ij} \cos(\theta_j - \theta_i)$$

for $i = j$ and

$$J_{ij} = \frac{\partial}{\partial \theta_j} f_i(\theta) = \gamma_{ij} \cos(\theta_j - \theta_i)$$

for $i \neq j$. Note that the row sums of J are all 0. If the system synchronizes, then the non-zeros eigenvalues J are all negative, and the rate of convergence of the system is related to the second largest eigenvalue.

Various Graphs and Interaction Functions

We investigated the case when the graph is K_n with uniform edge weights $\gamma > 0$ and the interaction function is the identity. In this case the system becomes

$$\frac{d}{dt}\vec{\theta}(t) = \gamma \begin{pmatrix} 1-n & 1 & \cdots & 1 \\ 1 & 1-n & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1-n \end{pmatrix} \vec{\theta}(t) + \vec{\omega}.$$

This has the particular solution

$$\vec{\theta}_p = T \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} - \frac{1}{n\gamma} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_n \end{pmatrix}$$

for any T since the components of $\vec{\omega}$ sum to zero. The eigenvalues of the matrix are 0 and $-n\gamma$ with multiplicities 1 and $n-1$ respectively and corresponding eigenvectors

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ \vdots \\ 0 \end{pmatrix}, \ldots \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ -1 \end{pmatrix}.$$

Therefore

$$\lim_{t\to\infty} \vec{\theta}(t) = T \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} - \frac{1}{n\gamma} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_n \end{pmatrix}$$

for some T which depends on the initial data since $\gamma > 0$.

Therefore for any vector ω whose components sum to 0 and $\gamma > 0$ the system synchonizes and has a unique steady state solution up to translation. Note that the rate of convergence increases as γ increases which makes sense intuitively since γ represents the strength of the interaction between the oscillators. Also notice that the factor $\frac{1}{n\gamma}$ shows that the difference between the phases of the oscillators in the steady state decreases as n and γ increase. This makes sense since as n increases the number of paths by which any two oscillators interact increases and since γ represents the strength of the interaction between the oscillators.

We now consider whether the oscillators synchronize on a general connected, undirected graph with positive edge weights for a given vector ω whose components sum to 0 with interaction function the identity. We then use our result to consider more general interaction functions

Theorem 1. *Let G be a connected, undirected graph with positive edge weights and $\vec{\omega}$ be an arbitrary vector whose components sum to 0, then the Kuramoto system with interaction function the identity, synchronizes to a unique steady state solution up to a translation for any initial data.*

Proof. Let $L(G)$ be the graph Laplacian of G, then the Kuramoto system becomes

$$\frac{d}{dt}\vec{\theta}(t) = L(G)\vec{\theta}(t) + \vec{\omega}.$$

We first demonstrate that the system has a costant particular solution $\vec{\theta_p}$. Such a solution satisfies

$$L(G)\vec{\theta_p} = -\vec{\omega}$$

hence we must show that $\vec{\omega} \in Im(L(G))$. To demonstrate this, let $\vec{e}_1, \cdots, \vec{e}_n$ be the standard basis for R^n, then $L(G)\vec{e}_1, \cdots, L(G)\vec{e}_n$ spans $Im(L(G))$. Note that the components of the vectors $L(G)\vec{e}_1, \cdots, L(G)\vec{e}_n$ all sum to 0 since $L(G)$ is symmetric and by definition has all row sums equal to 0. Also, the nullspace of $L(G)$ is 1-dimensional generated by

$$\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

since G is connected so that $Im(L(G))$ is $n-1$-dimensional by the rank-nullity theorem. Therefore $Im(L(G))$ is the subset of R^n consisting of vectors whose components sum to 0 hence $\vec{\omega} \in Im(L(G))$. Note that $\vec{\theta_p}$ is unique up to a translation by

$$\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Now, the system will synchronize if

$$\lim_{t\to\infty} \vec{\theta}_h(t)$$

exists where $\vec{\theta}_h(t)$ is the solution of the homogenous equation

$$\frac{d}{dt}\vec{\theta}_h(t) = L(G)\vec{\theta}_h(t).$$

This follows since 0 is a simple eigenvalue of $L(G)$ and all other eigenvalues are negative since G is connected and has positive edge weigths. In fact,

$$\lim_{t\to\infty} \vec{\theta}_h(t) \propto \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Note that the rate of convergence is related to the second largest eigenvalue of $L(G)$. Therefore the system synchronizes for any initial data to a steady state which is unique up to translation by

$$\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

□

We now consider how this result affects systems with more general interaction functions. We consider systems with an interaction function $I : R \to R$ satisfying $I(0) = 0$ and $I'(0) > 0$. Now the linearization of I at 0 is a positive multiple of the identity, and this multiplier can be incorporated into the edge weights so that we can assume that the identity is the linearization of I at 0. Let $(-\delta, \delta)$ be an interval such that the identity reasonably approximates I. Thus if $\theta_i(t^*) - \theta_j(t^*) \in (-\delta, \delta)$ for all i and j, then the systems

$$\frac{d}{dt}\theta_i(t) = \omega_i + \sum_j \gamma_{ij} I(\theta_i(t) - \theta_j(t))$$
$$\frac{d}{dt}\theta_i(t) = \omega_i + \sum_j \gamma_{ij} (\theta_i(t) - \theta_j(t))$$

should behave similarly near t^*. For this to hold true for a steady state we must have that

$$\omega_i = -\sum_j \gamma_{ij}(\theta_i - \theta_j)$$

for every i with $\theta_i - \theta_j \in (-\delta, \delta)$. Hence the range of $\|\omega\|_\infty$ for which the system synchronizes ought to increase with δ. Below we include an examle conjecture for this phenomenon.

Example. *Consider the one parameter family of interaction functions* $\{\lambda \sin\left(\frac{\theta}{\lambda}\right)\}$, *then* $\delta_\lambda \propto \lambda$ *hence the range of* $\|\omega\|_\infty$ *for which the system synchronizes ought to increase at least linearly in* λ.