Consensus Algorithms

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Introduction

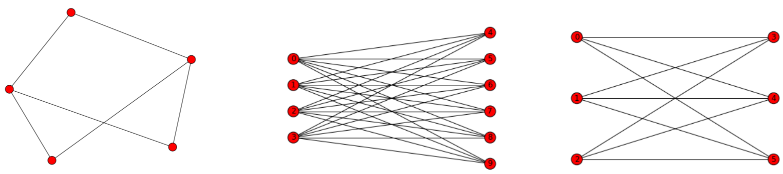
The physical application of this project is to understand the discrete heat flow equations. We viewed the rate of convergence, the Graph Laplacian spectrum, and edge weights. We first explored a planar graph with 12 edges and 8 vertices. Our goal was to find the edge weights that minimized the largest nonzero eigenvalue while maintaining the constraint that the sum of the edge weights must be zero. We did this by using the minimize function in the scipy.optimize package. This paper will address the uniqueness and structure of minimizers and classified the simple classes of graphs. As a result, we created a few conjectures.

Properties of Weight Distributions

Uniquenss

To determine if a graph had a unique optimal choice for weighted paths we wrote our program to minimize the largest nonzero eigenvalue and ran several iterations of this using randomly selected initial weights. We found graphs with multiple choices of weighted paths all of which gave the same largest nonzero eigenvalue. The examples we found were complete bipartite graphs of the form K_{mn} where $m \neq n$ (see figures 1a and 1b). We found that bipartite graphs always produced a unique optimal Laplacian matrix if and only if each node was connected to the same number of other nodes (see figure 1c).

A physical interpretation of the graphs is at each node we have a temperature and the paths our conductors where the number associated with the path determines the conductivity. The higher the weight the higher quick the nodes want to average their temperatures, while a negative weight would result in the temperatures repelling in value. Given our weights for the graph we can determine our Laplacian, L, so we have the relationship $\vec{x}_t$=L$\vec{x}$. This ode gives us the solution $\vec{x}$(t)=$e^{Lt}\vec{x}(0)$. Since L=XDX^T were D is the diagonal of L's eigenvalues and X is a matrix whose columns are the corresponding eigenvectors and X is orthogonal, we have that $\vec{x}$(t) $= Xe^{Dt}X^T\vec{x}(0)$. Now the key to convergence the heat of our nodes is that e^{Dt} converges as t approaches infinity; this happens if and only if all the eigenvalues are negative, it is not hard to see why this is true.

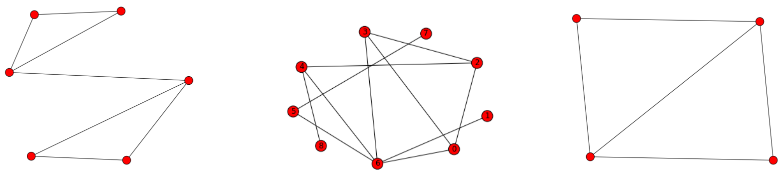


(a) K_{23} (b) K_{46} (c) K_{33}

Figure 1: Graphs w/o Unique Optimal Weights

Negative Gammas

Consider the constraint where the sum of the edges add to 1.0, and where we allow for negative weights. The graphs in figures 2a and 2b below are examples of graphs which have optimum eigenvalues while some of the weights are negative. In a physical system it may be hard to imagine a negative conductor of heat, but it is not hard to imagine that tactfully ordering the conductors is important and there are even cases were adding a conductor is less effective then reinforcing already existing conductors, see figure 2c. So it is not so unnatural to consider a case, given or constraints, where it makes sense to have negative conductors in order to greatly reinforce more important conductors, like a bottle neck.



(a) Two Connected Triangles (b) Graph that prefers Negative Weights (c) Graph with Weightless Edge

Figure 2: Graphs at Optimum State Prefer Negative Weights

Complete Graphs

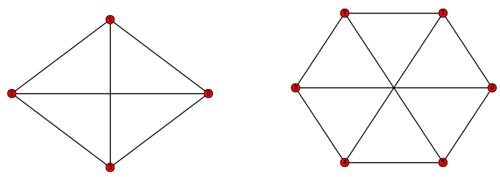
We first explored complete graphs. We quickly noticed that the weight distribution that minimized the largest non-zero eigenvalue was one where each edge had the same weight. We made a small conjecture stemming from that observation.

Conjecture 1 *Both the cyclic and complete graphs will end up with a uniform distribution of weights. The complete graph on n vertices will converge faster than the cyclic graph on n vertices.*

We conjectured that the complete graph will converge faster than the cyclic graph of the same size because the cyclic graph is part of the complete graph. The complete graph gives the most options for the weight distributions. One of those options would be putting zero weight on all the edges that aren't in one cycle through all the vertices. Our eigenvalues provided more evidence for this conjecture.

Cyclic Graphs

Conjecture 2 *When the number of nodes n are in the form $\{n : 2n \geq 2\}$ the Cube Symmetric Graphs (see figure 3) have a uniform weight distribution.*



(a) Case: 2(2)　　(b) Case: 2(3)

Figure 3: Pictures of the Cube Symmetric Graph

Other Types

We found that stars also had a minimum largest non-zero eigenvalue when the weight was distributed uniformly. The largest non-zero eigenvalue in this case is greater than that of the complete graph but less than that of the cyclic graph.

See Section for a discussion of bipartite graphs.

When we looked at graphs with less symmetric structure, the weights were distributed less uniformly.

Laplacian Spectrum

We looked at the Laplacian Spectrum of many different types of graphs to find a pattern in the eigenvalues of the Laplacian matrix.

Uniform Graphs

We explored many different graphs with fairly uniform structure. By uniform we mean that the graph can be drawn in a way that it looks symmetric. Additionally, these graphs have vertices of similar degrees with small exceptions.

Complete Graphs

We notice that the nonzero eigenvalues of the complete graphs we looked at were all equal. Additionally, the eigenvalues got close to zero as the number of nodes increased. That is not surprising because we would expect it to take longer for the weights to converge when there are more nodes and edges. See Table 1 for the weight each edge converges to and the eigenvalue of multiplicity N-1 for up to seven nodes.

N	weight	non-zero eigenvalue
3	$\frac{1}{3}$	-1
4	$\frac{1}{6}$	-0.666
5	$\frac{1}{10}$	-0.5
6	$\frac{1}{15}$	-0.4
7	$\frac{1}{21}$	-0.333

Table 1: Minimized Eigenvalue and Corresponding Weight

Cyclic Graphs

When we looked at cyclic graphs, we noticed that there were no more than two simple eigenvalues. In the first examples we looked at, the other eigenvalues had multiplicity two. After testing for three to seven nodes, we made a conjecture.

Conjecture 3 *The Laplacian of a cyclic graph will have no more than two eigenvalues of multiplicity one. All other eigenvalues will have multiplicity two.*

We tested larger cyclic graphs at random and we have yet to find a counter example.

Stars

The Laplacian of the stars we looked at had one very negative eigenvalue and one zero eigenvalue. All other eigenvalues were equal. Since the Laplacian has as many eigenvalues as there are vertices, this result should not be that surprising. The center vertex in a star (the one connected to all other vertices) has very different properties than the other vertices. That difference should be stored somehow in the spectrum of the Laplacian.

Other Graphs

We looked at graphs with less uniform structure to find patterns in the spectrum of the Laplacian. Besides learning that it is hard to create "random" graphs, we noticed some interesting things.

Unconnected Graphs

We explored graphs with two cycles that had no edges between them. For these graphs we noticed that the Laplacian had two eigenvalues that were zero. But the other eigenvalues had multiplicity one. This was the first graph we saw where the largest non-zero eigenvalue had multiplicity one.

Randomly Generated Graphs

We used the random graph generator that was built into networkx to explore more graphs. Though we did not have any large conjectures, we failed to find any graph that had a Laplacian with all simple eigenvalues.